MATH 1ZC3/1B03 Day Class: Final Exam - Version 1

Instructors: Bays, Buzano, Lozinski, McLean
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Duration: 3 hours

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Instructions:

This test paper contains 38 multiple choice questions printed on both sides of the page. The questions are on pages 2 through 21. Pages 22 to 26 are available for rough work. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF THE INVIGILATOR.

Select the one correct answer to each question and ENTER THAT ANSWER INTO THE SCAN CARD PROVIDED USING AN HB PENCIL. Room for rough work has been provided in this question booklet. You are required to submit this booklet along with your answer sheet. HOWEVER, NO MARKS WILL BE GIVEN FOR THE WORK IN THIS BOOKLET. Only the answers on the scan card count for credit. Each question is worth 1 mark. The test is graded out of 38. There is no penalty for incorrect answers. NO CALCULATORS are to be used in this exam.

Computer Card Instructions:

IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED. YOUR TEST RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.

The scanner that will read the answer sheets senses areas by their non-reflection of light. A heavy mark must be made, completely filling the circular bubble, with an HB pencil. Marks made with a pen or felt-tip marker will **NOT** be sensed. Erasures must be thorough or the scanner may still sense a mark. Do **NOT** use correction fluid.

- Print your name, student number, course name, and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.
- Mark your student number in the space provided on the sheet on Side 1 **and fill the corresponding bubbles underneath**.
- Mark only **ONE** choice (A, B, C, D, E) for each question.
- Begin answering questions using the first set of bubbles, marked "1".

1.

What are the solutions to

$$\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&5&5\end{bmatrix}\mathbf{x}=\begin{bmatrix}1\\2\\1\end{bmatrix}?$$

A) $\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}+t\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$ B) $\begin{bmatrix}0\\2\\-2\\1\end{bmatrix}+t\begin{bmatrix}2\\1\\1\\-2\end{bmatrix}$ [✓ C)] $\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}+t\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$

D) $\begin{bmatrix}1\\0\\1\\-2\end{bmatrix}+t\begin{bmatrix}-2\\4\\-2\\0\end{bmatrix}$ E) $\begin{bmatrix}0\\2\\-2\\1\end{bmatrix}+t\begin{bmatrix}-2\\1\\0\\-1\end{bmatrix}$

$$\left[\begin{array}{cccc:c}1&1&1&1&1\\1&2&3&4&2\\1&3&5&5&1\end{array}\right]\begin{array}{l}r_2\leftarrow r_2-r_1\\r_3\leftarrow r_3-r_1\end{array}\qquad\left[\begin{array}{cccc:c}1&1&1&1&1\\0&1&2&3&1\\0&2&4&4&0\end{array}\right]\begin{array}{l}r_1\leftarrow r_1-r_2\\r_3\leftarrow r_3-2r\end{array}$$

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}.\quad\left[\begin{array}{cccc:c}1&0&-1&-2&0\\0&1&2&3&1\\0&0&0&-2&-2\end{array}\right]r_1\leftarrow r_1-r_3\qquad\left[\begin{array}{cccc:c}1&0&-1&0&2\\0&1&2&3&1\\0&0&0&-2&-2\end{array}\right]r_3\leftarrow r_3\cdot-\tfrac{1}{2}$$

$$\left[\begin{array}{cccc:c}1&0&-1&0&2\\0&1&2&3&1\\0&0&0&1&1\end{array}\right]$$

$x_1 = x_3 + 2 = t+2$

$x_2 = -2x_3 - 3x_4 + 1 = -2x_3 - 2 = -2t-2$

$x_4 = 1$

$x_3 = t$

$$\vec{x}=\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}+\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}t$$

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2. For what value of k is the matrix A not invertible, where

$$A = \begin{bmatrix} k & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

A not invertible $\iff \det A = 0.$

A) 2 B) -1 [✓ C) -3] D) 4 E) -5

$$\det A = k \begin{vmatrix} 2 & 0 & 0 \\ 2 & 4 & 5 \\ 1 & 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 & 0 \\ 2 & 4 & 5 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= k \cdot 2 \begin{vmatrix} 4 & 5 \\ 1 & 1 \end{vmatrix} - 2(-1) \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 2k\,[4-5] + 2\,[2-5]$$

$$= -2k - 6 = 0 \iff 2k = -6 \iff k = -3.$$

3. A and B are invertible 3x3 matrices. det(B)=6 and

$$\det(2AB^{-1}) = \det I = 1.$$

What is $\det(A^2)$?

A) 1 B) 3 [✓ C) 9/16] D) 36 E) -9

$$1 = \det(2AB^{-1}) = 2^3 \det(A) \frac{1}{\det(B)} = 8 \det(A) \cdot \frac{1}{6} = \frac{4}{3} \det(A)$$

$$\Rightarrow \det(A) = 3/4.$$

$$\det(A^2) = \det(A \cdot A) = \det(A) \det(A) = 3/4 \cdot 3/4 = \frac{9}{16}.$$

Continued on page 4

4. Find the eigenvalue of A where

$$A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

The eigenvalue is:

[✓ A)] 4 B) 2 C) 3 D) 1 E) 5

$$\begin{vmatrix} 5-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)+1 = 15-8\lambda+\lambda^2+1$$

$$= \lambda^2-8\lambda+16 = (\lambda-4)(\lambda-4) \Rightarrow \lambda = 4.$$

5. Which of the following is not equivalent to the others for an nxn matrix A?

A) $Ax = 0$ has infinitely many solutions
B) 0 is an eigenvalue of A
C) $\dim(\text{null } A) > 0$
[✓ D)] There is at least one 0 on the main diagonal of A X
E) $\det(A) = 0$

e.g. $\Rightarrow A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ has $\det A = 0$

& $Ax = 0$ has inf. many solutions,
but not zero on main diagonal.

Continued on page 5

6. The 2x2 matrix A has eigenvalues 1 and -1, with corresponding eigenvectors $(4,1)$ and $(2,1)$. Which of the following could be the matrix A^3?

A) $\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$ B) $\begin{bmatrix} 2 & -6 \\ 3 & -3 \end{bmatrix}$ [✓ C)] $\begin{bmatrix} 3 & -8 \\ 1 & -3 \end{bmatrix}$

D) $\begin{bmatrix} 8 & 1 \\ 1 & -6 \end{bmatrix}$ E) $\begin{bmatrix} 64 & 8 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

$P = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}.$

$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}.$

$$A^3 = PD^3P^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^3 & 0 \\ 0 & (-1)^3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}\frac{1}{2}$$

$$= \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}\frac{1}{2} = \frac{1}{2}\begin{bmatrix} 6 & -16 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ 1 & -3 \end{bmatrix}.$$

7. Find a basis for the eigenspace associated with the eigenvalue $\lambda = 2$ for the matrix

$$A = \begin{bmatrix} 3 & -2 & -3 \\ -3 & 8 & 9 \\ 2 & -4 & -4 \end{bmatrix}$$

$Ax = 2x$

$(A - 2I)x = 0.$

A) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ B) $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$ C) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

[✓ D)] $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ E) $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$

$$\left[\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ -3 & 6 & 9 & 0 \\ 2 & -4 & -6 & 0 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 + 3r_1 \\ r_3 \leftarrow r_3 - 2r_1 \end{array} \quad \left[\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = 2y + 3z = 2t + 3s.$

$y = t$

$z = s$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} s.$$

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e.g.$\Rightarrow$ $A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$ invertible ($\det A = 9$).
$\lambda = 4$ double eigenvalue.

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$ $x = -t$, $y = t$ $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

only one eigenvector.

8. Which of the following statements are true?
i) If a matrix is invertible, it must be diagonalizable X
ii) If a matrix is diagonalizable, it must be invertible X
iii) Every nxn matrix with n different eigenvalues is diagonalizable ✓

e.g.$\Rightarrow$ $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
diagonalizable...
but <u>not</u> invertible.

A) i) only [✓ B) iii) only] C) ii) and iii)
D) i) and iii) E) All of these are true

9. A is a 3x3 matrix of rank 2. The system of equations

$$Ax = [3\ \ 5\ \ 7]^T$$

has infinitely many solutions, including $x = [1\ \ 2\ \ 3]^T$ and $x = [4\ \ 4\ \ 4]^T$.
A basis for the null space of A is:

A) $\{(1,1,1),(2,3,5)\}$ B) $\{(1,2,3),(3,5,7)\}$ [✓ C) $\{(3,2,1)\}$]
D) $\{(3,5,7)\}$ E) $\{(2,3,4)\}$

$Ax = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

$A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

$A\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

So, $A\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - A\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\| $

$A\left(\begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}\right) \Rightarrow \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} t \in \text{nullspace}(A)$, for any t.

$3 = \underbrace{\text{rank} A}_{2} + \text{nul}(A)$

$\Rightarrow \text{nul}(A) = 1$.

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$\left\{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right\}$.

10. For what value of k are the following polynomials linearly dependent?

$$1 + 3x^3 \qquad x - 2x^3 \qquad 5 + 2x + kx^3$$

A) 7 B) 1 C) -12 D) 0 [✓ E) 11]

$$(1 + 3x^3)c_1 + (x - 2x^3)c_2 + (5 + 2x + Kx^3)c_3 = 0$$

$$(c_1 + 5c_3) + (c_2 + 2c_3)x + (3c_1 - 2c_2 + Kc_3)x^3 = 0$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 3 & -2 & K \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \det(A) = \begin{vmatrix} 1 & 2 \\ -2 & K \end{vmatrix} + 5\begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix}$$

$$= K + 4 + 5(-3)$$

$$= K - 11 = 0 \Rightarrow K = 11.$$

11. Consider the 3 vectors in M_{22},

$$M_1 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$\dim M_{22} = 4.$

Which of the following statements is correct?

A) The matrices are not linearly independent, and form a basis for M_{22} X (need 3 things at least)
B) The matrices are linearly independent, and form a basis for M_{22} X
C) The matrices are not linearly independent, and do not form a basis for M_{22}
[✓ D)] The matrices are linearly independent, and do not form a basis for M_{22} ✓

$$K_1M_1 + K_2M_2 + K_3M_3 = 0$$

$$\begin{bmatrix} 3K_1 + 2K_3 & 3K_1 + K_2 \\ 3K_1 + K_2 & 3K_1 + 5K_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$3K_1 + 2K_3 = 0$
$3K_1 + K_2 = 0$
$3K_1 + K_2 = 0$
$3K_1 + 5K_3 = 0$

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 \end{array}\right] \quad \begin{array}{l} r_3 \leftarrow r_3 - r_2 \\ r_4 \leftarrow r_4 - r_1 \end{array} \quad \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$4K_3 = 0 \Rightarrow K_3 = 0$
$3K_1 + K_3 = 0$
$\Rightarrow 3K_1 = 0$
$\Rightarrow K_1 = 0.$

$3K_1 + K_2 = 0$
$\Rightarrow K_2 = 0.$

$\therefore \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ only solution $\Rightarrow$ lin. independent.

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12. In $\mathbb{R}^3$, consider the following vectors

$$\mathbf{v}_1 = (h, 1, 0), \quad \mathbf{v}_2 = (4, 1, h), \quad \mathbf{v}_3 = (1, -1, -3),$$

where $h \in \mathbb{R}$. For which values of h does the equation $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3$ hold?

A) $h = 0$ B) $h = 2$ [✓ C)] $h = 1$

D) $h = -2$ E) all h

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ h & 1 & 0 \\ 4 & 1 & h \end{vmatrix}$$

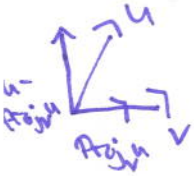
$$= i\begin{vmatrix} 1 & 0 \\ 1 & h \end{vmatrix} - j\begin{vmatrix} h & 0 \\ 4 & h \end{vmatrix} + k\begin{vmatrix} h & 1 \\ 4 & 1 \end{vmatrix}$$

$$= hi - h^2 j + (h-4)k$$

$$= (h, -h^2, h-4) = (1, -1, -3)$$

$$\Rightarrow h = 1.$$

13. Let $\mathbf{v} = (1, 1, 0)$ and $\mathbf{u} = (0, 3, 1)$ be two vectors in $\mathbb{R}^3$. Find two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ such that $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$.



A) $\mathbf{w}_1 = \frac{1}{2}(3, 3, 2)$ and $\mathbf{w}_2 = \frac{1}{2}(-3, 3, 0)$ [✓ B)] $\mathbf{w}_1 = \frac{1}{2}(3, 3, 0)$ and $\mathbf{w}_2 = \frac{1}{2}(-3, 3, 2)$

C) $\mathbf{w}_1 = \frac{1}{2}(3, 3, 0)$ and $\mathbf{w}_2 = (-3, 3, 2)$ D) $\mathbf{w}_1 = (3, 3, 0)$ and $\mathbf{w}_2 = \frac{1}{2}(-3, 3, 2)$

E) $\mathbf{w}_1 = (3, 3, 0)$ and $\mathbf{w}_2 = (-3, 3, 2)$

$$w_1 = \text{Proj}_v u = \frac{u \cdot v}{\|v\|^2} v = \frac{0+3+0}{(\sqrt{1+1})^2}(1,1,0) = \frac{3}{2}(1,1,0) = \left(\frac{3}{2}, \frac{3}{2}, 0\right).$$

$$u - \text{Proj}_v u = (0,3,1) - \left(\frac{3}{2}, \frac{3}{2}, 0\right) = \left(\frac{-3}{2}, \frac{3}{2}, 1\right).$$

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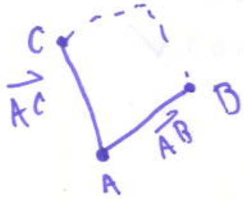
14. In $\mathbb{R}^3$, what is the area of the triangle with vertices $\underbrace{(1,1,1)}_{A}$, $\underbrace{(0,0,0)}_{B}$ and $\underbrace{(0,0,1)}_{C}$?



A) 3 [✓ B) $\frac{1}{\sqrt{2}}$] C) $\sqrt{3}$ D) 1 E) $\frac{\sqrt{3}}{2}$

$$\vec{AB} = (-1,-1,-1).$$
$$\vec{AC} = (-1,-1,0).$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{vmatrix}$$
$$= -i - j(-1) + k(0)$$
$$= (-1, 1, 0).$$

$$\tfrac{1}{2}\|\vec{AB} \times \vec{AC}\| = \tfrac{1}{2}\sqrt{(-1)^2 + 1^2}$$
$$= \tfrac{1}{2}\sqrt{2} = \tfrac{1}{\sqrt{2}}.$$

15. Let $\mathbf{u} = (1,1,1)$ and $\mathbf{v} = (0,1,0)$ be two vectors in $\mathbb{R}^3$. What are all the vectors $\mathbf{w} \in \mathbb{R}^3$ that lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$, are orthogonal to $\mathbf{v}$ and have unit norm?

[✓ A) $\frac{1}{\sqrt{2}}(1,0,1)$ and $\frac{1}{\sqrt{2}}(-1,0,-1)$ only] B) $(0,0,0)$ only

C) There are no such vectors D) $\frac{1}{2}(1,0,1)$ and $\frac{1}{2}(-1,0,-1)$ only

E) $\frac{1}{\sqrt{6}}(1,2,1)$ only

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -i - 0j + k = (-1,0,1).$$

$-x + z = 0$ is the plane u & v lie in.

$$\text{proj}_v u = \frac{1}{1}(0,1,0) = (0,1,0).$$

$$u - \text{proj}_v u = (1,1,1) - (0,1,0) = (1,0,1).$$

$$\|(1,0,1)\| = \sqrt{1^2+1^2} = \sqrt{2}.$$

$$\underbrace{\pm\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right).}$$

orthog. to v, length 1,
& can see lies in $-x+z$ plane.

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16. Let A be a 3×3 matrix with only one eigenvalue λ with algebraic multiplicity 3. Which of the following statements is true?

→ i.e.) λ is a triple root.

- i) A is always diagonalizable. X
- ii) A is diagonalizable if and only if the eigenspace corresponding to λ has dimension 3. ✓
- iii) A can never be diagonalizable. X

A) i) and iii) only B) ii) and iii) only C) i) only
D) i), ii) and iii) [✓ E)] ii) only

17. Let A be an $m \times n$ matrix. Which of the following statements is true?

→ i.e.) rankA = n.

- i) The column space of A is a subspace of $\mathbb{R}^n$. X
- ii) If $m = n$, then the row space and the column space of A are both $\mathbb{R}^n$ if and only if A is invertible. ✓
- iii) The dimensions of the row space and of the column space are always the same. ✓ (Theorem in book).

A) i), ii) and iii) B) ii) only C) i) and iii) only
D) i) only [✓ E)] ii) and iii) only

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18. Which of the following statements are always true:

i) $\frac{1}{i} = -i$ ✓

ii) $\arg(\bar{z}) = -\arg(z)$ ✓

iii) $\cos(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2}$ $= i\sin\theta$. X

(i) $\frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$. ✓

(ii) $z = e^{i\theta}$. $\bar{z} = e^{-\theta i}$. So, $\arg(\bar{z}) = -\theta = -\arg(z)$. ✓

A) i) is true, ii) and iii) are not

B) ii) and iii) are true, i) is not

[✓ C)] i) and ii) are true, iii) is not

D) None of the statements are true

E) All three are true

(iii) $e^{i\theta} = \cos\theta + i\sin\theta$.

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$.

$$\frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2}$$

$$= \frac{2i\sin\theta}{2} = i\sin\theta. \ X$$

19. Let

$$A = \begin{bmatrix} 1 & 5 & 2 & 6 & 7 \\ 1 & 0 & 2 & 1 & 7 \\ 1 & 3 & 2 & 4 & 7 \end{bmatrix} \begin{matrix} \\ r_2 \leftarrow r_2 - r_1 \\ r_3 \leftarrow r_3 - r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 5 & 2 & 6 & 7 \\ 0 & -5 & 0 & -5 & 0 \\ 0 & -2 & 0 & -2 & 0 \end{bmatrix} r_3 \leftarrow r_3 - \frac{2r}{5}$$

What is the dimension of the row space of A?

[✓ A)] 2 B) 4 C) 1 D) 5 E) 3

$$\begin{bmatrix} 1 & 5 & 2 & 6 & 7 \\ 0 & -5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 \leftarrow r_2 \cdot -\frac{1}{5} \\ r_1 \leftarrow r_1 + r_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 7 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

rank $A = 2$.

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20. Let W be the set of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 + x_4 = 0$. If we are considering the usual addition and scalar multiplication of $\mathbb{R}^4$, which of the following statements is true?

[✓ A)] W is closed under addition and scalar multiplication, therefore it is a subspace. ✓
B) W does not contain the vector $(1, 0, 0, 1)$, therefore it cannot be a subspace.
C) W is not closed under addition, therefore it is not a subspace.
D) W is not closed under scalar multiplication, therefore it is not a subspace.
E) W contains the zero vector, therefore it is not a subspace.

$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, x_1 + x_4 = 0 \right\}$. lex $x, y \in W \Rightarrow x_1 + x_4 = 0$ & $y_1 + y_4 = 0$.

$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix}$. $(x_1 + y_1) + (x_4 + y_4) = \underbrace{(x_1 + x_4)}_{\overset{\shortparallel}{0}} + \underbrace{(y_1 + y_4)}_{\overset{\shortparallel}{0}} = 0.$ ✓

$kx = \begin{pmatrix} kx_1 \\ kx_2 \\ kx_3 \\ kx_4 \end{pmatrix}$. $kx_1 + kx_4 = k\underbrace{(x_1 + x_4)}_{\overset{\shortparallel}{0}} = k * 0 = 0.$ ✓

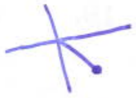
21. A student is given a complex number, and asked to find the fourth roots. If one of the roots is given by:

$$1 - \sqrt{3}i$$

which of the following must also be a root?

$r = (1^2 + \sqrt{3}^2)^{\frac{1}{2}} = 2.$

$2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

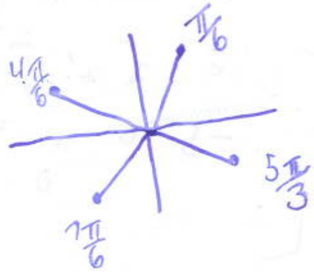


$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

A $-1 - \sqrt{3}i$ [✓ B) $-\sqrt{3} - i$] C) -1 D) $-2 + 3i$ E) $1 + \sqrt{3}i$

$1 - \sqrt{3}i$ a 4th root $\Rightarrow z = (1 - \sqrt{3}i)^4 = \left(2e^{\frac{5\pi}{3}}\right)^4$.

We know the roots must be evenly spaced: (i.e. $\frac{\pi}{2}$ a part from each other).

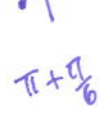


$\frac{5\pi}{3} = \frac{10\pi}{6}$.

$\frac{\pi}{2} = \frac{3\pi}{6}$

Also they must have the same radius.

$-\sqrt{3} - i = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = 2e^{7\frac{\pi}{6}}$. ✓



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$-1 - \sqrt{3}i = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2e^{4\frac{\pi}{3}i}$. X

$\pi + \frac{\pi}{6} =$

$\pi + \frac{\pi}{3} = 4\frac{\pi}{3}$

22. Suppose A and B are skew-symmetric invertible square matrices. Which of the following statements must hold?
(Recall that a matrix M is skew-symmetric if and only if $M^T = -M$). $A = -A^T$

(i) A^{-1} is skew-symmetric ✓
(ii) A^{-1} is symmetric X
(iii) If $AB = BA$, then AB is skew-symmetric X
(iv) If $AB = BA$, then AB is symmetric ✓

(i) $(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -A^{-1}$ ✓

(iii) & (iv) $(AB)^T = B^T A^T = (-B)(-A) = BA = AB$
$\Rightarrow AB$ symmetric.

A) (i) only B) (ii) and (iv) only C) (ii) and (iii) only
D) (ii) only [✓ E)] (i) and (iv) only

23. Divide:

$$\frac{1-3i}{2+i}\,\frac{(2-i)}{(2-i)} = \frac{2-7i-3}{4+1} = \frac{-1-7i}{5} = -\frac{1}{5}-\frac{7}{5}i.$$

A) $-\frac{1}{2} - 3i$ B) $-2 - 3i$ C) $5 - 3i$ [✓ D)] $-\frac{1}{5} - \frac{7}{5}i$ E) $-1 - 7i$

Continued on page 14

24. Gordon eats his lunch at any one of 3 different restaurants: Al's Chinese, BurgerBoom, or Chicken Mountain. On day t, the probability that he eats at each is given by a_t, b_t, and c_t respectively. He never eats at the same place two times in a row. After he eats at Al's, he is equally likely to eat at BurgerBoom or Chicken Mountain the next time. Likewise after he eats at BurgerBoom, he is equally likely to eat at Al's or Chicken Mountain the next time. If a dynamical system for (a_t, b_t, c_t) has a steady state vector of $\underset{x}{(6/15, 4/15, 5/15)}$, what is the probability that after he eats at Chicken Mountain, he eats at Al's the next time?

A) 2/3 B) 1/3 C) 1/2 [✓ D) 4/5] E) 1/10

$$\begin{array}{ccc} a & b & c \end{array} \quad M$$

$$M = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & \cancel{P_{13}} \\ 0.5 & 0 & \cancel{P_{23}} \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

$P_{11} = 0$
$P_{22} = 0$
$P_{33} = 0$

$P_{21} = 0.5$
$P_{31} = 0.5.$

$P_{12} = 0.5.$
$P_{32} = 0.5.$

Want P_{13}.

We know $M[x] = x$.

$$Mx = \begin{bmatrix} 0 & \frac{1}{2} & P_{13} \\ \frac{1}{2} & 0 & P_{23} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 6/15 \\ 4/15 \\ 5/15 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{30} + \frac{5P_{13}}{15} \\ * \\ * \end{bmatrix} = \begin{bmatrix} 6/15 \\ 4/15 \\ 5/15 \end{bmatrix}$$

$\Rightarrow \frac{4}{30} + \frac{5}{15} P_{13} = \frac{6}{15}$

$\Rightarrow \frac{1}{3} P_{13} =$ [illegible] $\frac{6}{15} - \frac{2}{15} = \frac{4}{15}$

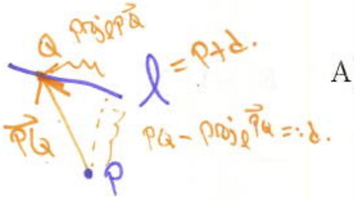
$\Rightarrow P_{13} = \frac{4}{15} \cdot 3 = \frac{12}{15} = \frac{4}{5}.$

25. What point on the line

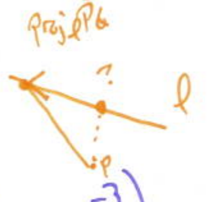
$$\ell = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$P =$

is closest to the point (2, 2, 2)?



A) (−1/3, 1, 1) B) (−1, 1, −1) [✓ C) (1/5, 1, 13/5)]
D) (1/2, 1, 7/2) E) (1, 1, 5)



$Q = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in \ell.$

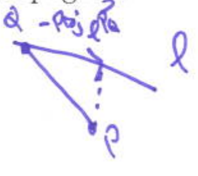
$\vec{PQ} = (-2, -1, 0).$

$$\text{proj}_\ell \vec{PQ} = \frac{\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}}{1^2 + 3^2} (1,0,3) = \frac{-2}{10} (1,0,3) = -\frac{1}{5}(1,0,3) = \left(\frac{-1}{5}, 0, \frac{-3}{5}\right).$$

$-\text{proj}_\ell \vec{PQ} = \left(\frac{1}{3}, 0, \frac{3}{5}\right).$

$Q + (-\text{proj}_\ell \vec{PQ}) = \left(\frac{1}{3}, 1, \frac{13}{[illegible]}\right)$

Continued on page 15



26. Suppose V is a vector space of dimension 3, and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are elements of V. Suppose $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = V$, and $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0$. Which of the following are bases of V?

dim3 ⇒ need 3 things in a basis.

- $S_0 := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ ✗
- $S_1 := \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ ✓
- $S_2 := \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ ✓
- $S_3 := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ ✓
- $S_4 := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ ✗

$v_1 + v_2 - v_3 = 0 \Rightarrow \{v_1, v_2, v_3\}$ lin. dependent set.

$\therefore v_3 \in \text{span}\{v_1, v_2\} \Rightarrow \text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3\}$

$\Rightarrow \text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_4\}$. ✓

Similarly, $v_1 \in \text{span}\{v_2, v_3\} \Rightarrow \text{span}\{v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_3, v_4\}$. ✓

Similarly,
$v_2 \in \text{span}\{v_1, v_3\}$
$\Rightarrow \text{span}\{v_1, v_3, v_4\} = V$. ✓

A) S_1, S_2, S_3, S_4 only B) S_3 only C) S_2, S_3, S_4 only

[✓ D)] S_1, S_2, S_3 only E) S_0 only

27. What is the dimension of the following subspace of the vector space M_{22} of 2x2 real matrices:

$$W := \text{span}\left(\left\{\underbrace{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}}_{v_3}\right\}\right)?$$

A) 3 [✓ B)] 2 C) 1 D) It has infinite dimension. E) 4

$v_1 + v_2 = v_3 \Rightarrow \text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_2\} = W$.

v_1 & v_2 lin. independent, b/c not differ by scalar multiple.

$\therefore \dim W = 2$.

Continued on page 16

28. Let $B = (\overbrace{x^2 + x + 1}^{v_1}, \overbrace{x^2 + x - 1}^{v_2}, \overbrace{x^2 - x + 1}^{v_3})$, which is an ordered basis of the vector space P_2 of polynomials of degree at most 2. What is the co-ordinate vector with respect to B of the quadratic polynomial $(x + 1)^2$?

A) $\begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix}$ B) $\begin{bmatrix} 0 \\ 1/2 \\ -3/2 \end{bmatrix}$ C) $\begin{bmatrix} 0 \\ 1/2 \\ -3/2 \end{bmatrix}$

D) $\begin{bmatrix} -1/2 \\ 0 \\ 3/2 \end{bmatrix}$ [✓ E)] $\begin{bmatrix} 3/2 \\ 0 \\ -1/2 \end{bmatrix}$

$(x+1)^2 = (x^2+2x+1)$

$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \\ -1/2 \end{pmatrix}.$

$K_1 v_1 + K_2 v_2 + K_3 v_3 = x^2+2x+1$

$\Rightarrow (K_1+K_2+K_3)x^2 + (K_1+K_2-K_3)x + (K_1-K_2+K_3) = x^2+2x+1$

$\Rightarrow K_1+K_2+K_3 = 1$
$K_1+K_2-K_3 = 2$
$K_1-K_2+K_3 = 1$

$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 - r_1 \\ r_3 \leftarrow r_3 - r_1 \end{array} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & -2 & 0 & 0 \end{array}\right]$

$K_2 = 0$
$-2K_3 = 1 \Rightarrow K_3 = -1/2.$
$K_1 = 1 - K_2 - K_3$
$= 1 - 0 + 1/2$
$= 3/2$

29. Let M_{22} be the set of 2x2 invertible matrices, with the usual scalar multiplication and a new vector addition operation given by the following:

If $\mathbf{u} = A$, $\mathbf{v} = B$ in M_{22}, then $\mathbf{u} + \mathbf{v} = AB$.

The fourth axiom of real vector spaces states that there must exist a unique element, $\mathbf{0}$, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in our set. What is this $\mathbf{0}$ for our given system?

A) No such matrix B) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ [✓ C)] I

D) A^{-1} E) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$u = A.$
$\bar{0} = B.$

$u \oplus \bar{0} = u$
$\Rightarrow AB = A$
$\Rightarrow B = I.$

Continued on page 17

30. Let A be the matrix

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A}.$$

Find A^{-1}.

A) $\begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ B) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ [✓ C)] $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

D) $\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ E) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \begin{matrix} r_1 \leftarrow r_1 - r_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Start.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] r_2 \leftarrow r_2 - r_1 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] r_2 \leftarrow r_2 + r_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$r_1 \leftarrow r_1 - r_2$
$r_3 \leftarrow r_3 - r_2$

31. What value appears in row 1, column 2 of the matrix obtained by the product AB where

$$A = \begin{bmatrix} 2+3i & 2 \\ 0 & 1-7i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5-7i & 1-i \\ 0 & 3i \end{bmatrix}$$

A) $3-3i$ B) $2-2i$ [✓ C)] $5+7i$ D) $4+10i$ E) $3-i$

$$AB = \begin{bmatrix} 2+3i & 2 \\ 0 & 1-7i \end{bmatrix} \begin{bmatrix} 5-7i & 1-i \\ 0 & 3i \end{bmatrix} = \begin{bmatrix} \ast & 2-2i+3i+3+6i \\ \ast & \ast \end{bmatrix}$$

$$= \begin{bmatrix} \ast & 5+7i \\ \ast & \ast \end{bmatrix}.$$

Continued on page 18

32. Given the vectors:

$$\mathbf{u}_1 = (-1, 2, 0)$$
$$\mathbf{u}_2 = (7, -9, -1)$$
$$\mathbf{u}_3 = (10, 0, 2)$$

The Gram-Schmidt process is used on this basis of $\mathbb{R}^3$ to produce an orthogonal basis. The first two vectors are:

$$\mathbf{v}_1 = (-1, 2, 0)$$
$$\mathbf{v}_2 = (2, 1, -1)$$

What is the third vector in the new basis?

A) $(-2, 1, 3)$ B) $(10/15, 0, 2/15)$ C) $(-3, 1, 1)$
[✓ D)] $(2, 1, 5)$ E) $(6, 3, 0)$

$$V_3 = u_3 - \text{proj}_{v_2} u_3 - \text{proj}_{v_1} u_3 = (10,0,2) - \frac{20-2}{4+1+1}(2,1,-1) - \frac{-10}{1+4}(-1,2,0)$$

$$= (10,0,2) - \frac{18}{6}(2,1,-1) + 2(-1,2,0) = (10,0,2) - (6,3,-3) + (-2,4,0)$$

$$= (2,1,5).$$

33. Let W be the subspace of $\mathbb{R}^4$ spanned by the orthogonal vectors: $\{\overbrace{(0, 1, 1, 1)}^{v_1}, \overbrace{(1, 1, 0, -1)}^{v_2}\}$ Compute the orthogonal projection of the vector $\mathbf{u} = (2, 1, 2, 0)$ onto this subspace.

A) $(3, 6, 3, 0)$ [✓ B)] $(1, 2, 1, 0)$ C) $(2, 0, 1, -1)$
D) $(-1, -5, -1, 0)$ E) $(3, 5, -2, 1)$

$$\text{proj}_W u = \text{proj}_{v_1} u + \text{proj}_{v_2} u = \frac{3}{3}(0,1,1,1) + \frac{3}{3}(1,1,0,-1)$$

$$= (0,1,1,1) + (1,1,0,-1) = (1,2,1,0).$$

Continued on page 19

34. Which of the following **IS** an orthogonal set of vectors, but is **NOT** orthonormal?

[✓ A)] $\left\{\overbrace{\frac{1}{\sqrt{2}}(0,1,0,-1)}^{v_1}, \overbrace{\frac{1}{\sqrt{2}}(0,1,0,1)}^{v_2}, \overbrace{\frac{1}{\sqrt{2}}(2,0,2,0)}^{v_3}\right\}$

B) $\left\{\frac{1}{2}(0,\sqrt{3},0,1), \frac{1}{2}(0,1,0,\sqrt{3}), \frac{1}{2\sqrt{2}}(2,0,2,0)\right\}$

C) $\left\{\frac{1}{5}(0,3,4,0), \frac{1}{5}(4,0,0,3), \frac{1}{5}(-3,0,0,4)\right\}$

D) $\left\{(1,0,0,0), \frac{1}{\sqrt{2}}(0,1,1,0), \frac{1}{\sqrt{2}}(0,0,1,1)\right\}$

E) $\left\{\frac{1}{4}(0,3,4,0), \frac{1}{4}(4,0,0,3), \frac{1}{4}(3,0,0,4)\right\}$

$$\left.\begin{aligned} v_1\cdot v_2 &= 0 \\ v_1\cdot v_3 &= 0 \\ v_2\cdot v_3 &= 0 \end{aligned}\right\} \Rightarrow \text{orthogonal.}$$

But $\|v_3\| = \sqrt{\left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{2}{\sqrt{2}}\right)^2}$

$= \sqrt{\frac{4}{2} + \frac{4}{2}} = \sqrt{\frac{8}{2}} = 2 \Rightarrow v_3$

<u>not</u> a unit vector.

35. Given the complex number

$$z = -2 + 2i \qquad \bar{z} = -2 - 2i.$$

write its complex conjugate in polar form.

A) $2\sqrt{2}e^{-i\pi/4}$ [✓ B)] $2\sqrt{2}e^{5i\pi/4}$ C) $2\sqrt{2}e^{-5i\pi/4}$. D) $-2e^{-i\pi/4}$ E) $-2e^{i\pi/4}$

$\pi + \frac{\pi}{4} = \frac{5\pi}{4}.$

$r = \sqrt{4+4} = \sqrt{8} = \sqrt{4\cdot 2} = 2\sqrt{2}.$

$\bar{z} = 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$\bar{z} = 2\sqrt{2}\, e^{\frac{5\pi}{4}i}.$

Continued on page 20

36. Which of the following commands used directly in the MatLab workspace produces a function that corresponds to:

$$f(x) = \begin{cases} x^2 & x > 2 \\ 1 - x & x \leq 2 \end{cases}$$

[✓ A)] `f= @(x) (x>2)*x^2+(x<=2)*(1-x)`

B) `f = {x^2 1-x x=2 }`

C) `f:= (x)-> case[(>2)(x^2) (<=2)(1-x)`

D) `f= @(x) (x>2)=x^2 (x<=2)=(1-x)`

E) `f:= (x)->if(x>2) x^2 else(1-x)`

37. Let V be a vector space and let W_1 and W_2 be two subspaces. Which of the following statements is always true?

i) The subset $W_1 \cap W_2$ of all vectors $\mathbf{v} \in V$ such that $\mathbf{v} \in W_1$ and $\mathbf{v} \in W_2$ is a subspace. ✓

ii) The subset $W_1 \cup W_2$ of all vectors $\mathbf{v} \in V$ such that $\mathbf{v} \in W_1$ or $\mathbf{v} \in W_2$ (or both) is a subspace. ✗

iii) $W_1 \cap W_2$ defined as in i) is never empty. ✓

A) ii) only [✓ B)] i) and iii) only C) i), ii) and iii) D) ii) and iii) only E) i) only

(i) $W_1 \cap W_2 = \{ v \in V \mid v \in W_1$ & $v \in W_2 \}$. Let $v, u \in W_1 \cap W_2$. Then $v + u \in W_1$ & $v + u \in W_2$ b/c both closed under "+" $\Rightarrow$ $v + u \in W_1 \cap W_2$. Similarly, $kv \in W_1$ & $kv \in W_2 \Rightarrow kv \in W_1 \cap W_2$. ✓

(ii) Let $v \in W_1 \cup W_2$ & $u \in W_1 \cup W_2$. Then $v + u$ may <u>not</u> be in $W_1 \cup W_2$. <u>e.g.:</u> If $v \in W_1$ but $v \notin W_2$ & $u \in W_2$, but $u \notin W_1$, then we don't know where $v+u$ will land. <u>e.g.:</u> union of two lines... add elements together not nec. land on either line.

Continued on page 21

(iii) $0 \in W_1$ & $0 \in W_2$ b/c vector subspaces of $V \Rightarrow 0 \in W_1 \cap W_2$.

38. Suppose A is a 3x3 matrix and

$$A\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

(i) Nontrivial solution to homogeneous system of equations $\Rightarrow$ $Ax=b$ <u>not</u> consistent for some b.

Which of the following statements must hold?

(i) For some $\mathbf{b}$, $A\mathbf{x}=\mathbf{b}$ has no solution for $\mathbf{x}$. ✓

(ii) For all $\mathbf{b}$, $A\mathbf{x}=\mathbf{b}$ has no solution for $\mathbf{x}$. X For $b=\begin{bmatrix}0\\0\\0\end{bmatrix}$, has solution $x=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

(iii) For all $\mathbf{b}$, either $A\mathbf{x}=\mathbf{b}$ has no solution for $\mathbf{x}$ or it has infinitely many solutions for $\mathbf{x}$. Yes, b/c r.r.e.f. A has a row of zeros $\Rightarrow$ a free parameter if a solution exists. ✓

A) (i) and (ii) only B) (i), (ii), and (iii) C) (iii) only

D) (i) only [✓ E)] (i) and (iii) only

END OF TEST QUESTIONS

Continued on page 22

Extra page for rough work. DO NOT DETACH!

Continued on page 23

Math 1B03

Sample Exam

Name: DeDieu Lauren
(Last Name) (First Name)

Student Number:______________ **Tutorial Number:**______________

This exam consists of 40 multiple choice questions worth 1 mark each (no part marks), and 1 question worth 1 mark (no part marks) on proper computer card filling. All questions must be answered on the COMPUTER CARD with an HB PENCIL. Marks will not be deducted for wrong answers (i.e., there is no penalty for guessing). You are responsible for ensuring that your copy of the test is complete. Bring any discrepancy to the attention of the invigilator. Calculators are NOT allowed.

1. Let $\mathbf{p} = 2 - x + x^2$. Find the coordinates of $\mathbf{p}$ with respect to the following basis of P_2 $\{1+x, 1+x^2, x+x^2\}$.
(a) $(1,-1,2)$ [✓ **(b)** $(0,2,-1)$] **(c)** $(0,2,2)$ **(d)** $(2,-1,0)$ **(e)** $(-1,1,3)$ (See Paper)

2. Let V be a vector space with dimension n. Consider the following statements.
(i) Every independent set in V is a basis for V $\rightsquigarrow$ may not span. ✗
(ii) Every set in V that spans V must be independent $\rightsquigarrow$ only need n to span V... anymore then can span, but will <u>not</u> be independent. ✗
(iii) Every set in V with less than n vectors must be independent. ⤷ No! Take 2 vectors that are a multiple of each other. ✗
Which of the above statements is always true?
(a) (ii) and (iii) only **(b)** (iii) only **(c)** (ii) only [✓ **(d)** none of them] **(e)** (i) only

3. Find the dimension of the following vector spaces.
(i) The set of all 2×2 skew-symmetric matrices
(ii) The set of all polynomials $a + bx + cx^2$ where $a = b + c$.
[✓ **(a)** 1 and 2] **(b)** 2 and 3 **(c)** 1 and 3 **(d)** 3 and 3 **(e)** 4 and 2

(i) 2×2 skew-symmetric matrices have the form $A = -A^T \Rightarrow \begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d\end{bmatrix}$
$\Rightarrow a = -a \Rightarrow 2a = 0 \Rightarrow a = 0$
$\Rightarrow d = -d \Rightarrow 2d = 0 \Rightarrow d = 0.$
$\Rightarrow b = -c$. So, has a basis $\left\{\begin{bmatrix} 0 & 1 \\ & 0\end{bmatrix}\right\}$.

(ii)

4. If A is a 4×4 matrix and the columns of A are linearly dependent then,
(a) every vector $\mathbf{b}$ in $\mathbb{R}^4$ is in the column space of A
(b) no vector $\mathbf{b}$ is in the column space of A
(c) The column vectors of A form a basis for $\mathbb{R}^4$
[✓ **(d)** None of the above]

$A = \begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix}$ s.t. $\{v_1, v_2, v_3, v_4\}$ is a dependent set of vectors. (c) We know a basis of $\mathbb{R}^4$ must span $\mathbb{R}^4$ & be independent. ✗ (a) We know we need at least 4 independent vectors to span $\mathbb{R}^4$. ✗ (b) $v_1 \in CS(A)$, since $A\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = v_1$. ✗

5. Let $\mathbf{u} = (1,-2,1,6)$ in $\mathbb{R}^4$, and let $W = \text{span}\{(1,1,-1,0),(1,1,0,0)\}$. Compute $\text{proj}_W \mathbf{u}$.
(a) $(-\frac{1}{2}, 0, 1, \frac{1}{2})$ **(b)** $(-1, -\frac{1}{2}, \frac{1}{2}, 0)$ **(c)** $(-\frac{1}{2}, -1, 1, 0)$
[✓ **(d)** $(-\frac{1}{2}, -\frac{1}{2}, 1, 0)$] **(e)** $(\frac{1}{2}, -1, -\frac{1}{2}, 0)$

First, we need to find an orthogonal basis for $\text{span}\{v_1, v_2\}$.

$u_1 = (1,1,-1,0).$
$u_2 = v_2 - \text{Proj}_{u_1} v_2 = (1,1,0,0) - \frac{2}{3}(1,1,-1,0) = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0)$. $W = \text{span}\left\{\begin{pmatrix}1\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{2}{3}\\0\end{pmatrix}\right\}$.

Now that we have an orthogonal basis, $\text{Proj}_W u = \text{Proj}_{u_1} u + \text{Proj}_{u_2} u = \frac{u \cdot u_1}{\|u_1\|} u_1 + \frac{u \cdot u_2}{\|u_2\|} u_2$
$= \frac{-2}{3}(1,1,-1,0) + \frac{\frac{1}{3}}{\frac{6}{9}}(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0) = (-\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 0) + (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, 0) = (-\frac{1}{2}, -\frac{1}{2}, 1, 0).$

6. Find a basis of the following subspace of $\mathbb{R}^4$.

$W =$ all vectors of the form (a, b, c, d) where $a + b - c + d = 0$.

[✓ **(a)**] $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, 1)\}$
(b) $\{(1, 0, 0, -1), (0, 1, 0, -1)\}$
(c) $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 1, -1, 0)\}$
(d) $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 1, -1, 0)\}$
(e) $\{(1, 0, -1, 0), (0, 1, 0, -1), (0, 0, 1, -1)\}$

$a+b-c+d=0$
$\Rightarrow a=-b+c-d$

$b=t, c=s, d=r$

$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}t + \begin{pmatrix}1\\0\\1\\0\end{pmatrix}s + \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}r$ could work....

Also $d=-a-b+c$. $a=t, b=s, c=r$.

$\underset{a}{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}}t + \underset{b}{\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}}s + \underset{c}{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}}r$ ← corresponds to (a)!

7. Find the dimension of the subspace of $\mathbb{R}^3$ spanned by the following set of vectors.
$\{(1, 5, 6), (2, 6, 8), (3, 7, -1), (4, 8, 12)\}$

(a) 1 **(b)** 2 [✓ **(c)**] 3 **(d)** 4 **(e)** 0 (See Paper)

8. Decode the message AOJX given that it is a Hill cipher with enciphering matrix

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

(I don't think you're responsible for cryptography questions).

(a) MATE **(b)** HILL **(c)** HELP **(d)** GOOD **(e)** MATH

9. Consider the following statements.

(i) Suppose that $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and that $A\mathbf{u}_i = \mathbf{b}$ for each i. If the vector $\mathbf{u}$ is in W then $A\mathbf{u} = \mathbf{b}$. $u \in W \Rightarrow u = k_1u_1 + \cdots + k_ku_k$ for $k_i \in \mathbb{R} \Rightarrow Au = Ak_1u_1 + \cdots + Ak_ku_k = k_1b + \cdots + k_kb \neq b$. X

(ii) Let W be the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ that are solutions to the equation $A\mathbf{x} = 0$. W is a subspace of $\mathbb{R}^n$. • Let $x, y \in W \Rightarrow Ax = 0$ & $Ay = 0$. $A(x+y) = Ax + Ay = 0 + 0 = 0 \Rightarrow x+y \in W \Rightarrow$ closed under "+".
• $A(\alpha x) = \alpha Ax = \alpha * 0 = 0 \Rightarrow \alpha x \in W \Rightarrow$ closed under $*$.
• $A*0 = 0 \Rightarrow$ nonempty. ✓

Which of the above statements is always true?

(a) (i) only [✓ **(b)**] (ii) only **(c)** (i) and (ii) **(d)** neither

10. Find a basis for the null space of A. $A = \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc} 1 & 2 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ -1 & 0 & -3 & 2 \end{array}\right] \end{array} \begin{array}{c} :0 \\ :0 \\ :0 \end{array}$ $r_1 \leftarrow r_1 + r_3$

$\left[\begin{array}{cccc:c} 0 & 2 & -4 & 6 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ -1 & 0 & -3 & 2 & 0 \end{array}\right]$ $r_1 \leftarrow r_1 - 2r_2$

$\left[\begin{array}{cccc:c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ -1 & 0 & -3 & 2 & 0 \end{array}\right]$ $y = 2z - 3w$
$x = -3z + 2w$
$z = t$
$w = s$

(a) $\{(1, 0, -1), (2, 1, 0)\}$
(b) $\{(0, 2, -3, 1), (1, 2, -3, 0)\}$
(c) $\{(1, 2, -1, 4), (0, 1, -2, 3)\}$
(d) $\{(-1, 2, 0, 3), (2, 1, 0, -3)\}$
[✓ **(e)**] $\{(2, -3, 0, 1), (-3, 2, 1, 0)\}$

$\begin{pmatrix}-3\\2\\1\\0\end{pmatrix}t + \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}s$ is the general solution $\Rightarrow \left\{\begin{pmatrix}-3\\2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}\right\}$ basis for nullspace.

11. Let A be a matrix with 4 rows and 7 columns. Then the column space of A

[✓ **(a)**] is a subspace of $\mathbb{R}^4$
(b) has dimension 4
(c) is equal to the column space of A^T
(d) none of the above

$A = 4\overset{7}{\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix}}$

Column space ↓

ⓐ The columns of A are 4×1 vectors $\Rightarrow \overset{\text{span}}{\{v_1, v_2, v_3, v_4\}}$ is a subspace of $\mathbb{R}^4$. ✓ ⓑ However, v_1, v_2, v_3 & v_4 may or may not be linearly independent, so we don't know what the dimension is. X
ⓒ The column space of A^T equals the row space of A, which is a subspace of $\mathbb{R}^7$, so they're not equal. X

For Questions 12-14, determine which of the following answers is correct for the given subset W of $\mathbb{R}^3$.

(a) W is a subspace

(b) W is closed under addition, but not closed under scalar multiplication

(c) W is closed under scalar multiplication, but not closed under addition

(d) W is not closed under scalar multiplication, and not closed under addition

12. $W =$ all vectors of the form $(a, 3b, c)$ where $a = c + 1$. $W = \left\{ \begin{pmatrix} a \\ 3b \\ c \end{pmatrix} \in \mathbb{R}^3 \mid a = c+1 \right\}$. Let $x, y \in W$

(a) **(b)** **(c)** [✓ **(d)**] $\Rightarrow x = \begin{pmatrix} a \\ 3b \\ c \end{pmatrix}$ & $y = \begin{pmatrix} d \\ 3e \\ f \end{pmatrix}$ for some $a, b, c, d, e, f \in \mathbb{R}$ & $a = c+1$, $d = f+1$.

$x+y = \begin{pmatrix} a+d \\ 3(b+e) \\ c+f \end{pmatrix}$ & $a+d = (c+1)+f+1 = c+f+2 \neq c+f+1$. ✗ $\alpha x = \begin{pmatrix} \alpha a \\ \alpha 3b \\ \alpha c \end{pmatrix} = \begin{pmatrix} \alpha a \\ 3(\alpha b) \\ \alpha c \end{pmatrix}$ & $\alpha a = \alpha(c+1) = \alpha c + \alpha \neq \alpha c + 1$. ✗

13. $W =$ all vectors of the form $(2a, -b^2, -c)$ let $x, y \in W \Rightarrow x = (2a, -b^2, -c)$

(a) [✓ **(b)**] **(c)** **(d)** & $y = (2d, -e^2, -f)$ for some $a, b, c, d, e, f \in \mathbb{R}$. $x+y = (2(a+d), -(b^2+e^2), -(c+f))$ $= (2(a+d), -(\sqrt{b^2+e^2})^2, -(c+f)) \in W$. ✓ $\alpha x = (2(\alpha a), -\alpha b^2, -(\alpha c))$, but if α is negative, then can't write αb^2 as a square. ✗

14. Let $\mathbf{b}$ be a nonzero vector in $\mathbb{R}^4$ and let A be a 4×4 matrix.
Let $W =$ all vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are solutions to the equation $A\mathbf{x} = \mathbf{b}$.

(a) **(b)** **(c)** [✓ **(d)**] Let $x, y \in W \Rightarrow Ax = b$ & $Ay = b$ for some fixed b. $A(x+y) = Ax + Ay$ $= b + b = 2b \neq b$. ✗ $A(\alpha x) = \alpha Ax = \alpha b \neq b$. ✗

15. Suppose that W is a subspace of a vector space V. Consider the following statements.

(i) If $\mathbf{u}$ is in W and $a\mathbf{u} - b\mathbf{v}$ is in W (where $b \neq 0$) then $\mathbf{v}$ is in W. $au = -bv \Rightarrow v = -\frac{a}{b}u$ (since $b \neq 0$)

(ii) If $\mathbf{u}$ is in W and $\mathbf{v}$ is in W then $a\mathbf{u} - b\mathbf{v}$ is in W. $\Rightarrow v$ can be written as a linear combo. of vectors from $W \Rightarrow v \in W$. ✓ $u, v \in W \Rightarrow$ any linear combo. of u & v is in $W \Rightarrow (au - bv) \in W$. ✓

Which of the above statements are always true?

(a) (i) only

(b) (i) only

[✓ **(c)**] (i) and (ii)

(d) neither of them

16. Let $W = \text{span}\{\overset{u_1}{(1,1,1,1)}, \overset{u_2}{(3,1,3,1)}, \overset{u_3}{(6,2,4,0)}\}$. Find an orthonormal basis of W using the Gram-Schmidt process.

[✓ **(a)**] $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$

(b) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, -\frac{1}{2}, 1, -\frac{1}{2}), (\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})\}$

(c) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}})\}$

(d) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0), (0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$

(e) $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0), (0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})\}$

$v_1 = (1,1,1,1)$.

$v_2 = u_2 - \text{proj}_{v_1} u_2 = (3,1,3,1) - \frac{8}{4}(1,1,1,1)$
$= (1,-1,1,-1)$

$v_3 = u_3 - \text{proj}_{v_1} u_3 - \text{proj}_{v_2} u_3 = (6,2,4,0)$
$- (1,1,1,1) * \frac{12}{4} - (1,-1,1,-1)\frac{8}{4}$

$= (6,2,4,0) - (3,3,3,3) - (2,-2,2,-2) = (1,1,-1,-1)$.

So, $\{v_1, v_2, v_3\}$ is an orthogonal basis.

$\|v_1\| = \sqrt{4} = 2$. $\|v_2\| = \sqrt{4} = 2$. $\|v_3\| = \sqrt{4} = 2$.

So, $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$ is an orthonormal basis of W.

17. Consider the following set of orthogonal vectors,

$$\mathbf{v}_1 = (1, -1, 2, -1),\ \mathbf{v}_2 = (-2, 2, 3, 2),\ \mathbf{v}_3 = (1, 2, 0, -1),\ \mathbf{v}_4 = (1, 0, 0, 1).$$

Let $\mathbf{u} = (3, 1, -2, 4)$. Find c such that $\mathbf{u} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 + d\mathbf{v}_4$

(a) $\frac{5}{6}$ [✓ **(b)** $\frac{1}{6}$] **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{2}$ **(e)** $\frac{2}{3}$

(see Page)

18. If A and B are both $n \times n$ invertible matrices, which of the following matrices is the inverse of $(A^{-1}B)^T$?

[✓ **(a)** $(B^{-1}A)^T$] **(b)** $(AB^{-1})^T$ **(c)** $B^T(A^T)^{-1}$ **(d)** $(A^T)^{-1}B^T$ **(e)** $(B^TA^T)^{-1}$

$(A^{-1}B)^T = B^T(A^{-1})^T = B^T(A^T)^{-1} \Rightarrow ((A^{-1}B)^T)^{-1} = (B^T(A^T)^{-1})^{-1} = ((A^T)^{-1})^{-1}(B^T)^{-1} = A^T(B^{-1})^T$

$= (B^{-1}A)^T$.

19. Consider the following statements.

(i) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal in $\mathbb{R}^3$ then $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$

(ii) $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 + \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$.

Which of the above statements is always true?

(a) neither [✓ **(b)** (i) only] **(c)** (ii) only **(d)** (i) and (ii)

(i) $\|u+v\| = \sqrt{(u+v)\cdot(u+v)} = \sqrt{u\cdot u + \underbrace{u\cdot v}_{0} + \underbrace{v\cdot u}_{0} + v\cdot v} = \sqrt{u\cdot u + v\cdot v}$.

$\|u-v\| = \sqrt{(u-v)\cdot(u-v)} = \sqrt{u\cdot u - \underbrace{u\cdot v}_{0} - \underbrace{v\cdot u}_{0} + v\cdot v} = \sqrt{u\cdot u + v\cdot v}$. ✓

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(ii) $\frac{1}{4}\|u+v\|^2 + \frac{1}{4}\|u-v\|^2 = \frac{1}{4}(u\cdot u + 2u\cdot v + v\cdot v) + \frac{1}{4}(u\cdot u - 2u\cdot v + v\cdot v) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 \neq \|u\|^2 + \|v\|^2$. ✗

20. Recall that B is *similar* to A if there is an invertible matrix P such that $B = P^{-1}AP$. Suppose that B is similar to A. Consider the following statements.

(i) A and B have the same determinant

(ii) B^{-1} is similar to A^{-1}

(i) $B = P^{-1}AP \Rightarrow \det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(A)\det(P) = \det(A)$. ✓

(ii) $B = P^{-1}AP \Rightarrow B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$. ✓

Which of the above statements are always true?

(a) (i) only

(b) (ii) only

[✓ **(c)** (i) and (ii)]

(d) neither of them

21. A matrix P is called **orthogonal** if $PP^T = I$. Consider the following statements.

(i) If P is an orthogonal matrix then $2P$ is also orthogonal.

(ii) If P is an orthogonal matrix then $\det P = \pm 1$

Which of the above statements is always true?

(a) (i) ony [✓ **(b)** (ii) only] **(c)** (i) and (ii) **(d)** neither

(i) P orthog. $\Rightarrow PP^T = I$. $(2P)(2P)^T = 4PP^T = 4I \neq I$. ✗

(ii) $\det(PP^T) = \det(I) \Rightarrow \det(P)\det(P^T) = 1$

$\Rightarrow \det(P)\det(P) = 1 \Rightarrow (\det(P))^2 = 1$

$\Rightarrow \det(P) = \pm 1$. ✓

22. Let $A = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{bmatrix}$. Find the characteristic polynomial $p(\lambda)$ of A.

(a) $(\lambda+b)(\lambda+a)^2$ **(b)** $(\lambda-b)(\lambda+a)^2$ **(c)** $(\lambda-b)(\lambda-a)^2$

[✓ **(d)** $(\lambda-b)(\lambda-a)(\lambda+a)$] **(e)** $(\lambda+b)(\lambda-a)(\lambda+a)$

$p(\lambda) = \begin{vmatrix} -\lambda & 0 & a \\ 0 & b-\lambda & 0 \\ a & 0 & -\lambda \end{vmatrix} = b-\lambda \begin{vmatrix} -\lambda & a \\ a & -\lambda \end{vmatrix} = (b-\lambda)(\lambda^2 - a^2) = (b-\lambda)(\lambda+a)(\lambda-a) = (\lambda-b)(\lambda+a)(\lambda-a).$

23. Consider the following statements.

(i) $\{(1,-1,2,3),\ (2,1,-1,1),\ (1,\ 8,\ -13,-12)\}$ is an independent set.

(ii) $\{(1,2,-1),\ (-1,1,2),(-5,-1,8)\}$ spans $\mathbb{R}^3$.

Which of the above statements is true?

(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) [✓ **(d)** neither] (See Paper)

24. Consider the triangle with vertices P, Q, and R. Which of the following is a right-angled triangle?

(a) $P(1,1,0), Q(1,0,1), R(1,-1,2)$ **(b)** $P(1,1,0), Q(1,0,1), R(1,2,2)$

(c) $P(1,1,0), Q(1,0,1), R(1,0,2)$ **(d)** $P(1,1,0), Q(1,0,1), R(1,1,3)$

[✓ **(e)** $P(1,1,0), Q(1,0,1), R(1,3,2)$]

(see paper)

25. Find the shortest distance from the point $P(0,1,-1)$ to the line $(x,y,z) = (1,1,0) + t(1,-1,-2)$.

[✓ **(a)** $\frac{1}{6}\sqrt{66}$] **(b)** $\frac{1}{6}\sqrt{65}$ **(c)** $\frac{4}{3}$ **(d)** $\frac{1}{6}\sqrt{62}$ **(e)** $\frac{1}{6}\sqrt{61}$ (See Paper)

26. Find the equation of the plane containing the point $P(3,0,-1)$ and the line $(x,y,z) = (2,1,3) + t(3,-1,-2)$.

(a) $2x - 6y + 2z = 4$ [✓ **(b)** $x + 5y - z = 4$] **(c)** $x + 6y - z = 4$ (See Paper)

(d) $3x - 17y + 5z = 4$ **(e)** $16y - 4z = 4$

27. Consider the following matrix (where only the first row is given): $A = \begin{bmatrix} 3 & -2 \\ * & * \end{bmatrix}$. $\vec{x}$ eigenvector of A $\Leftrightarrow A\vec{x} = \lambda\vec{x}$ for some λ.

If $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ is an eigenvector of A, what is the corresponding eigenvalue?

(a) $2-2i$ **(b)** $2-i$ [✓ **(c)** $1+2i$] **(d)** $1-i$ **(e)** $3+i$

$\begin{bmatrix} 3 & -2 \\ * & * \end{bmatrix}\begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \begin{bmatrix} 3+3i-4 \\ * \end{bmatrix} = \begin{bmatrix} -1+3i \\ * \end{bmatrix} = \lambda\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$\Rightarrow -1+3i = \lambda(1+i) \Rightarrow \lambda = \frac{-1+3i(1-i)}{1+i\,(1-i)} = \frac{-1+i+3i+3}{1+1} = \frac{2+4i}{2} = 1+2i.$

28. Consider the line through $P(1,2,3)$ that is parallel to $\mathbf{v} = (1,0,1)$. Which of the following planes does the line lie in?

(a) $x + 2y + 2z + 1 = 0$ [✓ **(b)** $3x + 2y - 3z + 2 = 0$] **(c)** $-2y - z + 1 = 0$

(d) $3x - y + z + 2 = 0$ **(e)** $2x + 2y + z - 3 = 0$ (See Paper)

29. If A and B are $n \times n$ symmetric matrices, which of the following matrices are always symmetric?

(i) $A - B^T$

(ii) $A^TB - B^TA$

[✓ **(a)** (i) only] **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

A & B symmetric $\Rightarrow A = A^T$ & $B = B^T$. (i) $(A-B^T)^T = A^T - B^{T^T} = A^T - B = A - B = A - B^T$. ✓ (ii) $(A^TB - B^TA)^T = (A^TB)^T - (B^TA)^T = B^TA - A^TB \neq A^TB - B^TA$. ✗

$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - 2r_1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$E_2 E_1 A = B \Rightarrow U = E_2 E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$

30. Consider the following matrices.

$$A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

B can be obtained from A by the following sequence of row operations on A:
1. Switch row 1 and row 2
2. Replace row 2 by (row 2 − 2 × row 1)
Using the above sequence of row operations (in the above order), find an invertible matrix U such that $UA = B$.

(a) $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ [✓ **(b)**] $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$ **(d)** $\begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$ **(e)** $\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

31. Given that $\det \begin{bmatrix} 1 & a & -1 \\ 3 & -b & 1 \\ 3 & c & 4 \end{bmatrix} = 3$, solve the following system of equations for the variable y.

$$\begin{aligned} x + ay - z &= 2 \\ 3x - by + z &= 5 \\ 3x + cy + 4z &= 0 \end{aligned}$$

$\underbrace{\begin{bmatrix} 1 & a & -1 \\ 3 & -b & 1 \\ 3 & c & 4 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{3} \text{adj}(A)$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3} \text{adj}(A) \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \Rightarrow y = \frac{1}{3}\left[2C_{12} + 5C_{22}\right]$

$= \frac{1}{3}\left[-2[12-3] + 5[4+3]\right]$

$= -\frac{18}{3} + \frac{35}{3} = \frac{17}{3}$

[✓ **(a)**] $y = \frac{17}{3}$ **(b)** $y = \frac{8}{3}$ **(c)** $y = 5$ **(d)** $y = a - b + c$ **(e)** $y = 4a + 2c$

32. Compute the determinant of the following matrix.

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \end{bmatrix} C_2 \leftarrow C_2 + C_3$$

$\det \begin{bmatrix} 0 & 0 & -1 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 3 & 2 & 1 \\ 5 & 0 & 0 & 7 \end{bmatrix} = -3 \begin{vmatrix} 0 & -1 & 0 \\ 3 & 0 & 2 \\ 5 & 0 & 7 \end{vmatrix}$

(a) 0 **(b)** 5 [✓ **(c)**] −33 **(d)** −17 **(e)** 8 $= -3\left[+1\left[21-10\right]\right] = -33.$

33. Let A be a 2×2 matrix, with $\det A = 2$. Evaluate $\det(2\,\text{adj}(A))$.
(a) 2 **(b)** 4 [✓ **(c)**] 8 **(d)** 16 **(e)** 32

$\det(2\,\text{adj}(A)) = 2^2 \det(\text{adj}(A))$
$= 4 \det(\det(A) A^{-1}) = 4 \det(2A^{-1}) = 4 \cdot 2^2 \frac{1}{\det(A)} = \frac{16}{2} = 8.$

34. A square matrix P is called **idempotent** if $P^2 = P$. Let A and B be $n \times n$ idempotent matrices. Which of the following matrices are always idempotent?
(i) $A - B$
(ii) AB
(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) [✓ **(d)**] neither

We know $A^2 = A$ & $B^2 = B$.

(i) $(A-B)^2 = (A-B)(A-B) = A^2 - AB - BA + B^2 = A - AB - BA + B \neq A - B.$ ✗

(ii) $(AB)^2 = ABAB \neq AB.$ ✗

35. In a dynamical system for inheritance, suppose that the transition matrix has eigenvectors $\mathbf{x}_1 = (1, 2, 1)$, $\mathbf{x}_2 = (1, 0, -1)$, and $\mathbf{x}_3 = (1, -2, 1)$, with corresonding eigenvalues $\lambda_1 = 1, \lambda_2 = \frac{1}{2}$, and $\lambda_3 = 0$, respectively. If the initial state vector $\mathbf{v}_0$ can be written as $\mathbf{v}_0 = \frac{1}{4}\mathbf{x}_1 - \frac{1}{4}\mathbf{x}_2 + \frac{9}{50}\mathbf{x}_3$, find the constant b so that the state vector after 5 generations can be written as $\mathbf{v}_5 = a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3$.

(a) $-\frac{1}{1024}$ **(b)** $-\frac{1}{32}$ **(c)** $\frac{1}{32}$ [✓ **(d)** $-\frac{1}{128}$] **(e)** $\frac{1}{1024}$ (see Paper)

36. Let z be a complex number. Which of the following statements is correct? Let $z = a+bi$.

[✓ **(a)**] $\overline{z} + z$, $(\overline{z} - z)i$, $\overline{z}z$ are all real numbers ⓐ $(\bar z - z)i = (a-bi-a-bi)i = (-2bi)i = 2b$. ✓

(b) $\overline{z} + z$, $(\overline{z} - z)i$, $\overline{z}z$ all have modulus 1 $\bar z + z = a-bi+a+bi = 2a$. ✓ $\bar z z = (a-bi)(a+bi) = a^2+abi$

(c) $\overline{z} + z$ and $\overline{z}z$ are real numbers, but $(\overline{z} - z)i$ is not a real number. $-abi+b^2$

(d) If z is a complex number and $|z| = 1$, then $z = 1$ or $z = -1$. $= a^2+b^2$. ✓

(e) none of the above

37. Find all complex numbers z so that $z^3 = -8i$.

(a) $\sqrt{3} + i, -\sqrt{3} + i, -2i$ **(b)** $\sqrt{2} - i, -\sqrt{2} - i, 2i$ **(c)** $\sqrt{3} - i, -\sqrt{3} + i, -2i$

(d) $\sqrt{2} + i, -\sqrt{2} + i, 2i$ [✓ **(e)** $\sqrt{3} - i, -\sqrt{3} - i, 2i$] (see Paper)

38. Find a matrix P which diagonalizes

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4$

$= \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$

$\Rightarrow \lambda = 3, \lambda = -1.$

(a) $\begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$ [✓ **(c)** $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$] **(d)** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\underline{\lambda=1}$: $\left[\begin{array}{cc|c} & -2 & 0 \\ -2 & 2 & 0 \end{array}\right] r_1 \leftarrow r_1 + r_2$ $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -2 & 2 & 0 \end{array}\right]$ $2x = 2y \Rightarrow x = t.$ $y = t$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix} t$ gen. eigenspace.

$\underline{\lambda=3}$: $\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array}\right] r_1 \leftarrow r_1 - r_2$ $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -2 & -2 & 0 \end{array}\right]$ $x = -y$ $y = t$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix} s$ gen. eigenspace. $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

39. Let A be an $n \times n$ matrix. Suppose that there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix. Consider the following statements.

(i) $A^2 = P^2D^2(P^{-1})^2$ (i) We know $A = PDP^{-1} \Rightarrow A^2 = PD\underbrace{P^{-1}P}_{I}DP^{-1} = PD^2P^{-1} \neq P^2D^2(P^{-1})^2$. ✗

(ii) $A^2 = P^{-1}D^2P$ (ii)

Which of the above statements is always true?

(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) [✓ **(d)**] neither

40. The arrival of a bus at a particular stop can be classified as either an early arrival or a late arrival. If it is early on one day, then there is a 60% chance that it will be early the next day. If it is late on one day, then there is a 90% chance that it will be late the next day. In the long run, what proportion of times is the bus late?

(a) $\frac{2}{5}$ [✓ **(b)** $\frac{4}{5}$] **(c)** $\frac{3}{5}$ **(d)** $\frac{7}{10}$ **(e)** $\frac{9}{10}$

$T = \begin{array}{c|cc|c} & & & \\ \hline & p_{11} & p_{12} & E \\ & p_{21} & p_{22} & L \\ \hline & E & L & \end{array}$ p_{ij} is prob. system moves from state j to state i.

$p_{11}: (E \to E) = 0.6$

$p_{12}: (L \to E) = 0.1$

$p_{21}: (E \to L) = 0.4$

$p_{22}: (L \to L) = 0.9$

So $T = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$.

Must find steady-state vector (i.e. $\lambda = 1$ eigenvector):

$\left[\begin{array}{cc|c} 0.6-1 & 0.1 & 0 \\ 0.4 & 0.9-1 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -0.4 & 0.1 & 0 \\ 0.4 & -0.1 & 0 \end{array}\right] r_1 \leftarrow r_1 + r_2$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0.4 & -0.1 & 0 \end{array}\right]$ $\frac{4}{10}x = \frac{1}{10}y \Rightarrow x = \frac{1}{4}t$

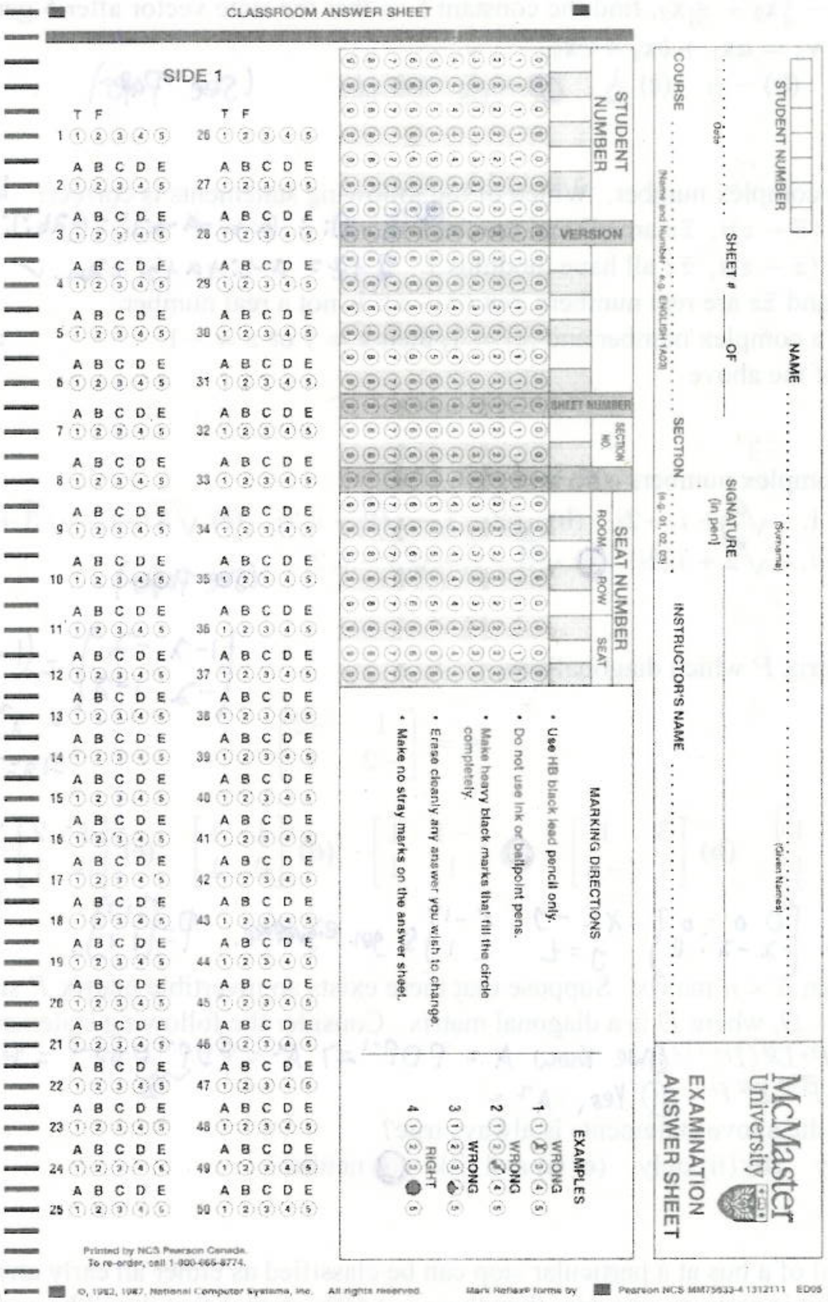
$y = t$ $\begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix} t$ eigenvector.

$\frac{1}{4}t + t = 1$

$\frac{5}{4}t = 1 \Rightarrow t = \frac{4}{5}$.

$\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$ steady state.

41. Correctly fill out the bubbles corresponding to your student number and the version number of your test in the correct places on the computer card. (Use the below computer card for this sample test.)



CLASSROOM ANSWER SHEET

SIDE 1

T F

1 ① ② ③ ④ ⑤ 26 ① ② ③ ④ ⑤

A B C D E

2 ① ② ③ ④ ⑤ 27 ① ② ③ ④ ⑤

3 ① ② ③ ④ ⑤ 28 ① ② ③ ④ ⑤

4 ① ② ③ ④ ⑤ 29 ① ② ③ ④ ⑤

5 ① ② ③ ④ ⑤ 30 ① ② ③ ④ ⑤

6 ① ② ③ ④ ⑤ 31 ① ② ③ ④ ⑤

7 ① ② ③ ④ ⑤ 32 ① ② ③ ④ ⑤

8 ① ② ③ ④ ⑤ 33 ① ② ③ ④ ⑤

9 ① ② ③ ④ ⑤ 34 ① ② ③ ④ ⑤

10 ① ② ③ ④ ⑤ 35 ① ② ③ ④ ⑤

11 ① ② ③ ④ ⑤ 36 ① ② ③ ④ ⑤

12 ① ② ③ ④ ⑤ 37 ① ② ③ ④ ⑤

13 ① ② ③ ④ ⑤ 38 ① ② ③ ④ ⑤

14 ① ② ③ ④ ⑤ 39 ① ② ③ ④ ⑤

15 ① ② ③ ④ ⑤ 40 ① ② ③ ④ ⑤

16 ① ② ③ ④ ⑤ 41 ① ② ③ ④ ⑤

17 ① ② ③ ④ ⑤ 42 ① ② ③ ④ ⑤

18 ① ② ③ ④ ⑤ 43 ① ② ③ ④ ⑤

19 ① ② ③ ④ ⑤ 44 ① ② ③ ④ ⑤

20 ① ② ③ ④ ⑤ 45 ① ② ③ ④ ⑤

21 ① ② ③ ④ ⑤ 46 ① ② ③ ④ ⑤

22 ① ② ③ ④ ⑤ 47 ① ② ③ ④ ⑤

23 ① ② ③ ④ ⑤ 48 ① ② ③ ④ ⑤

24 ① ② ③ ④ ⑤ 49 ① ② ③ ④ ⑤

25 ① ② ③ ④ ⑤ 50 ① ② ③ ④ ⑤

STUDENT NUMBER

VERSION

SHEET NUMBER

SECTION NO.

SEAT NUMBER

ROOM ROW SEAT

MARKING DIRECTIONS

- Use HB black lead pencil only.
- Do not use ink or ballpoint pens.
- Make heavy black marks that fill the circle completely.
- Erase cleanly any answer you wish to change.
- Make no stray marks on the answer sheet.

EXAMPLES

1 WRONG

2 WRONG

3 WRONG

4 RIGHT

STUDENT NUMBER

Date

COURSE .... (Name and Number - e.g. ENGLISH 1A03)

SHEET # .... OF ....

NAME .... (Surname) .... (Given Names)

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Math 1ZC3 - Sample Exam

1. $p = 2 - x + x^2$. $\{1+x, 1+x^2, x+x^2\}$.

So, we need to find an (a,b,c) s.t.
$a(1+x) + b(1+x^2) + c(x+x^2) = 2 - x + x^2$

$\Rightarrow a + ax + b + bx^2 + cx + cx^2 = 2 - x + x^2$
$\Rightarrow (a+b) + (a+c)x + (b+c)x^2 = 2 - x + x^2$
$\Rightarrow a+b=2$ & $a+c=-1$ & $b+c=1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ r_2 \leftarrow r_2 - r_1 \end{array} \left[\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} r_1 \leftarrow r_1 + r_2 \\ r_3 \leftarrow r_3 + r_2 \end{array} \left[\begin{array}{ccc:c} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

$x = -z - 1 = 1 - 1 = 0$ $(a,b,c) = (0, 2, -1)$.
$y = z + 3 = -1 + 3 = 2$
$2z = -2 \Rightarrow z = -1$

7. We want to Know the maximum number of vectors we can keep in our set s.t. the set is independent.

$$\left[\begin{array}{ccc:c} 1 & 5 & 6 & 0 \\ 2 & 6 & 8 & 0 \\ 3 & 7 & -1 & 0 \\ 4 & 8 & 12 & 0 \end{array}\right] \begin{array}{l} \\ r_2 \leftarrow r_2 - 2r_1 \\ r_3 \leftarrow r_3 - 3r_1 \\ r_4 \leftarrow r_4 - 4r_1 \end{array} \left[\begin{array}{ccc:c} 1 & 5 & 6 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & -8 & -19 & 0 \\ 0 & -12 & -12 & 0 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 * \frac{1}{4} \\ r_4 \leftarrow r_4 * \frac{1}{12} \end{array} \left[\begin{array}{ccc:c} 1 & 5 & 6 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -8 & -14 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$r_4 \leftarrow r_4 - r_2$
$r_3 \leftarrow r_3 - 8r_2$

$$\left[\begin{array}{ccc:c} 1 & 5 & 6 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The dimension of the nullspace is one $\Rightarrow$ the dimension of this subspace is 3.

17. $(3, 1, -2, 4) = av_1 + bv_2 + cv_3 + dv_4$

$$\left[\begin{array}{cccc:c} 1 & -2 & 1 & 1 & 3 \\ -1 & 2 & 2 & 0 & 1 \\ 2 & 3 & 0 & 0 & -2 \\ -1 & 2 & -1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ \\ r_2 \leftarrow r_2 + r_1 \\ r_4 \leftarrow r_4 + r_1 \end{array} \quad \left[\begin{array}{cccc:c} 1 & -2 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 4 \\ 2 & 3 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 7 \end{array}\right] r_1 \leftarrow r_1 - r_2$$

$$\left[\begin{array}{cccc:c} 1 & -2 & -2 & 0 & -1 \\ 0 & 0 & 3 & 1 & 4 \\ 2 & 3 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 7 \end{array}\right]$$

$3c = -d + 4 = -\frac{7}{2} + \frac{8}{2} = \frac{1}{2} \Rightarrow c = \frac{1}{6}.$

$2d = 7 \Rightarrow d = \frac{7}{2}$

23. (i) $\{v_1, v_2, v_3\}$ independent $\Leftrightarrow$ $av_1 + bv_2 + cv_3$ <u>only</u> has the solution $a = b = c = 0$.

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ -1 & 1 & 8 & 0 \\ 2 & -1 & -13 & 0 \\ 3 & 1 & -12 & 0 \end{array}\right] \begin{array}{l} \\ r_2 \leftarrow r_2 + r_1 \\ r_3 \leftarrow r_3 - 2r_1 \\ r_4 \leftarrow r_4 - 3r_1 \end{array} \quad \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & -5 & -15 & 0 \\ 0 & -5 & -15 & 0 \end{array}\right] \begin{array}{l} r_3 \leftarrow r_3 \times \frac{1}{5} \\ r_2 \leftarrow r_2 \times \frac{1}{3} \\ r_4 \leftarrow r_4 - r_3 \end{array}$$

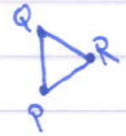
$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] r_3 \leftarrow r_3 - r_2 \quad \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3 columns, but only 2 leading 1's
$\Rightarrow$ 1 parameter
$\Rightarrow$ nontrivial solutions
$\Rightarrow$ <u>not</u> independent. ✗

(ii) $\{v_1, v_2, v_3\}$ span $\mathbb{R}^3 \Leftrightarrow \forall (a,b,c) \in \mathbb{R}^3 \; \exists (k_1, k_2, k_3) \in \mathbb{R}^3$ s.t. $k_1v_1 + k_2v_2 + k_3v_3 = (a,b,c)$. i.e. is $\underbrace{\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}}_{A} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ consistent.

Consistent $\Leftrightarrow \det(A) \neq 0$.

$$\begin{vmatrix} 1 & -1 & -5 \\ 2 & 1 & -1 \\ -1 & 2 & 8 \end{vmatrix} \begin{array}{l} \\ r_2 \leftarrow r_2 + r_1 \\ r_3 \leftarrow r_3 + r_1 \end{array} \quad \begin{vmatrix} 1 & -1 & -5 \\ 3 & 0 & -6 \\ 0 & 1 & 3 \end{vmatrix} = 1 \times 9 - 1 \times (-6 + 15) = 9 - 9 = 0.$$ ✗



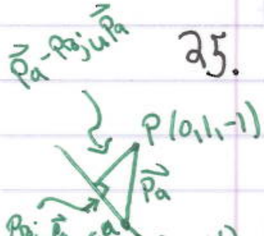
24. ⓐ $\vec{PQ} = (0,-1,1)$, $\vec{PR} = (0,-2,2)$, $\vec{QR} = (0,-1,1)$

$\vec{PQ}\cdot\vec{PR} = 2+2 = 4 \neq 0$. $\vec{PQ}\cdot\vec{QR} = 1+1 = 2 \neq 0$. $\vec{PR}\cdot\vec{QR} = 2+2 = 4 \neq 0$. X

We know a right-angle has a 90° angle $\Rightarrow$ one of the 2 vectors must have dot product zero.

ⓔ $\vec{PQ} = (0,-1,1)$, $\vec{PR} = (0,2,2)$, $\vec{QR} = (0,3,1)$.

$\vec{PQ}\cdot\vec{PR} = 0-2+2 = 0$. ✓



$l = \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{\text{Position}} + \underbrace{\begin{pmatrix}1\\-1\\-2\end{pmatrix}}_{\substack{u\\ \text{direction}}} t$

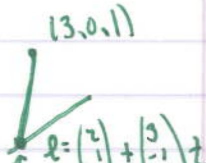
25. $\vec{Pa} = (1,0,1)$. $Proj_u\vec{Pa} = \frac{\vec{P_a}\cdot u}{\|u\|^2}u = \frac{-1}{6}(1,-1,-2) = (\frac{-1}{6}, \frac{1}{6}, \frac{1}{3})$.

$\vec{P_a} - Proj_u\vec{Pa} = (1,0,1) - (\frac{-1}{6}, \frac{1}{6}, \frac{1}{3}) = (\frac{7}{6}, \frac{-1}{6}, \frac{2}{3})$.

$\|\vec{P_a} - Proj_u\vec{Pa}\| = \sqrt{\frac{50}{36}+\frac{4}{9}} = \sqrt{\frac{50}{36}+\frac{16}{36}} = \underline{\frac{1}{6}\sqrt{66}}$.

$\overset{2}{16}$
$\frac{4}{64}$
$\overset{31}{4)\overline{124}}$
12
4

26. **Recall:** The Point normal eq'n of a plane is given by $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, where $P_0(x_0,y_0,z_0)$ is a specific point on the plane, & $n(a,b,c)$ is the normal vector to the plane.



$l = \begin{pmatrix}2\\1\\3\end{pmatrix} + \begin{pmatrix}3\\-1\\-2\end{pmatrix} t$

$\begin{pmatrix}2\\1\\3\end{pmatrix}$

$(3,0,-1)$ & $(2,1,3)$ are points on our plane & the line b/w these 2 points has direction $(1,-1,-4)$.

$\underbrace{(3,-1,-2) \times (1,-1,-4)}_{\text{gives the vector normal to these 2 lines.}} = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & -1 & -4 \end{vmatrix} = (4-2)i - (-12+2)j + (-3+1)k = (2,10,-2)$.

$0 = 2(x-3)+10(y-0)-2(z+1) = 2x-6+10y-2z-2$
$= 2x+10y-2z-8 \Rightarrow 2x+10y-2z = 8 \Rightarrow \underline{x+5y-z=4}$.

28. If $v = (1,0,1)$ is parallel to the plane, then $v\cdot n = 0$, where n is the normal vector to the plane. Let $n = (a,b,c)$.

$$v\cdot n = \begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix} = a + c = 0 \Rightarrow a = -c.$$

So, our normal vector should look like $n = (a,b,-a)$, & our plane should look like:

$$a(x-1) + b(y-2) - a(z-3) = 0$$
$$ax + by - az - a - 2b + 3a = 0$$
$$ax + by - az + 2a - 2b = 0.$$

We can see that ⓑ is the only option that works: $a=3$, $b=2$, $c=-3$:

$$3x + 2y - 3z + 6 - 4 = 0$$
$$3x + 2y - 3z + 2 = 0.$$

(You also could have tried each option: if it was the right option then plugging $(1,2,3)$ into it should make the eq^n equal zero, & when you dot its normal with v you get zero. e.g., For ⓐ $1 + 2(2) + 2(3) + 1 = 12 \neq 0$, so $x + 2y + 2z + 1 = 0$ does not contain $P(1,2,3)$. For ⓒ, $(0,-2,-1)\cdot(1,0,1) = -1 \neq 0$ so ⓒ fails).

35. Our transition matrix T has 3 distinct eigenvalues $\Rightarrow$ T is diagonalizable $\Rightarrow$ $T = PDP^{-1}$, where

$$P = \begin{bmatrix}1 & 1 & 1\\2 & 0 & -2\\1 & -1 & 1\end{bmatrix} \quad \& \quad D = \begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{2} & 0\\0 & 0 & 0\end{bmatrix}.$$

ha! I suppose this wasn't necessary... oh well.

$$\det(P) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{vmatrix} \; r_1 \leftarrow r_1 + r_3 \; = \begin{vmatrix} 2 & \boxed{0} & 2 \\ 2 & \boxed{0} & -2 \\ 1 & \boxed{-1} & 1 \end{vmatrix} = 1 * \begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix} = -8.$$

$$\operatorname{adj}(P) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} -2 & -4 & -2 \\ -2 & 0 & 2 \\ -2 & 4 & -2 \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & -2 \\ -4 & 0 & 4 \\ -2 & 2 & -2 \end{bmatrix}.$$

$$P^{-1} = \frac{1}{\det(P)} \operatorname{adj}(P) = \frac{-1}{8} \begin{bmatrix} -2 & -2 & -2 \\ -4 & 0 & 4 \\ -2 & 2 & -2 \end{bmatrix}.$$

So, $T = PDP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & -2 & -2 \\ -4 & 0 & 4 \\ -2 & 2 & -2 \end{bmatrix} -\frac{1}{8}.$

We know $v_0 = \frac{1}{4} x_1 - \frac{1}{4} x_2 + \frac{9}{50} x_3$

$$= \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{9}{50} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(1) - \frac{1}{4}(1) + \frac{9}{50}(1) \\ \frac{1}{4}(2) - \frac{1}{4}(0) + \frac{9}{50}(-2) \\ \frac{1}{4}(1) - \frac{1}{4}(-1) + \frac{9}{50}(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/4 \\ -1/4 \\ 9/50 \end{bmatrix} = P \begin{bmatrix} 1/4 \\ -1/4 \\ 9/50 \end{bmatrix}.$$

By def^n, we know $v_5 = T^5 v_0 = PD^5P^{-1} v_0 = PD^5 \underbrace{P^{-1}(P}_{I} \begin{bmatrix} 1/4 \\ -1/4 \\ 9/50 \end{bmatrix})$

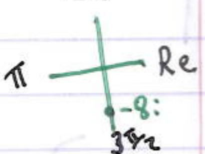
$$= PD^5 \begin{bmatrix} 1/4 \\ -1/4 \\ 9/50 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1^5 & 0 & 0 \\ 0 & (\frac{1}{2})^5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/4 \\ -1/4 \\ 9/50 \end{bmatrix}}_{\text{* these first}}$$

32
4
128

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/32 \cdot -\frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ -1/128 \\ 0 \end{bmatrix} = \frac{1}{4} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{x_1} - \frac{1}{128} \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{x_2} = \frac{1}{4} x_1 \boxed{-\frac{1}{128}} x_2 + 0 x_3.$$

32
28

37. First, let's put $-8i$ in polar form:



$-8i = re^{\theta i}$
$= r\cos\theta + r\, i\sin\theta$

$r = |-8i| = \sqrt{0^2+(-8)^2} = \sqrt{64} = 8.$
$\theta = \frac{3\pi}{2}.$

So, $-8i = 8e^{\frac{3\pi}{2}i}$.

Let $z = r_0 e^{\phi i}$.

So, $z^3 = -8i$ becomes $r_0^3 e^{3\phi i} = 8e^{\frac{3\pi}{2}i}$

$\Rightarrow r_0^3 = 8 \Rightarrow r_0 = 2$ & $\Rightarrow 3\phi = \frac{3\pi}{2} + 2k\pi$ for $k = 0, 1, 2$

$\Rightarrow \phi = \frac{\pi}{2} + \frac{2}{3}k\pi \Rightarrow \phi = \frac{\pi}{2}$ or $\frac{\pi}{2} + \frac{2\pi}{3}$ or $\frac{\pi}{2} + \frac{4\pi}{3}$

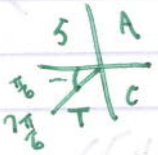
$\sin(\frac{\pi}{6}) = \frac{1}{2}$

$\Rightarrow \phi = \frac{\pi}{2}$ or $\frac{3\pi}{6} + \frac{4\pi}{6}$ or $\frac{3\pi}{6} + \frac{8\pi}{6} \Rightarrow \phi = \frac{\pi}{2}$ or $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

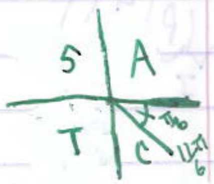
So, $z = 2e^{\frac{\pi}{2}i}$ or $2e^{\frac{7\pi}{6}i}$ or $2e^{\frac{11\pi}{6}i}$.

$\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

Converting these to rectangular form we get:



$2e^{\frac{\pi}{2}i} = 2(\cos\frac{\pi}{2} + i\sin(\frac{\pi}{2})) = \underline{2i}.$
$2e^{\frac{7\pi}{6}i} = 2(\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})) = 2(\frac{-\sqrt{3}}{2}) + 2i(\frac{-1}{2}) = \underline{-\sqrt{3} - i}.$
$2e^{\frac{11\pi}{6}i} = 2(\cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6})) = 2(\frac{\sqrt{3}}{2}) + 2i(\frac{-1}{2}) = \underline{\sqrt{3} - i}.$



$2\pi - \frac{11\pi}{6} = \frac{\pi}{6}.$