Lauren DeDieu

Sample Test 2

2. Solve the equation $z(1+i)+i=1$ in the complex numbers

A) $z = 1 + i$ B) $z = -i$ C) $z = i$ D) $z = 0$

E) No complex numbers z can satisfy the equation

$z(1+i)+i=1$

$z(1+i)=1-i$

$z = \frac{1-i}{1+i} \cdot \frac{(1-i)}{(1-i)}$

$z = \frac{1-2i-1}{1+1} = \frac{-2i}{2} = \boxed{-i}$

Sample Test 2

3. What is $e^{\pi i/6} + e^{5\pi i/6}$?

(A) i (circled) B) $1+i$ C) $\frac{i}{2}$ D) $1-i$ E) $-i$

$e^{\frac{\pi}{6}i} = \cos\frac{\pi}{6} + i\sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + i\frac{1}{2}.$

$e^{\frac{5\pi}{6}i} = \cos\frac{5\pi}{6} + i\sin(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}.$

So, $e^{\frac{\pi}{6}i} + e^{\frac{5\pi}{6}i} = \frac{\sqrt{3}}{2} + \frac{1}{2}i - \frac{\sqrt{3}}{2} + \frac{1}{2}i = \boxed{i}.$

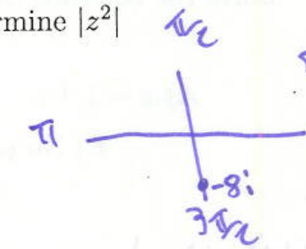
4. Suppose $z^3 = -8i$. What is $|z^2|$?

(A) 4 (circled) B) $-\sqrt{8}$ C) 2 D) $\sqrt{8}$

E) There is insufficient information to determine $|z^2|$

$z^3 = -8i.$ let $z = r_0 e^{\phi i}.$

$-8i = 8e^{\frac{3\pi}{2}i}.$



$r = |-8i| = 8.$

$r_0^3 e^{3\phi i} = 8e^{\frac{3\pi}{2}i}$

$\Rightarrow r_0^3 = 8 \Rightarrow r_0 = 2$

& $\Rightarrow 3\phi = 3\frac{\pi}{2}i + 2k\pi$

$\Rightarrow \phi = \frac{\pi}{2}i + \frac{2}{3}\pi k$ for $k = 0, 1, 2.$

Choose $k=0$:

$z = 2e^{\frac{\pi}{2}i}.$

$z^2 = 4e^{\pi i}.$

$\boxed{|z^2| = 4}.$

5. Let $\omega = e^{2i\pi/3}$. Note that ω is a cube root of unity, i.e. $\omega^3 = 1$. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Using the definition of an eigenvector, determine which of the following are complex eigenvectors of A.

$$(I)\ \underbrace{\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}}_{x} \qquad (II)\ \underbrace{\begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}}_{y}$$

A) Neither B) Just (I) C) Just (II) D) Both (I) and (II) (circled)

E) I am a fish

• $x \neq 0$ is an eigenvector if $Ax = \alpha x$ for some scalar α.

$$Ax = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{bmatrix} \omega \\ \omega^2 \\ 1 \end{bmatrix} = \omega \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \quad (\omega^3 = 1)$$

$\Rightarrow x$ is an eigenvector w/ eigenvalue ω.

$$Ay = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix} = \begin{bmatrix} \omega^2 \\ \omega \\ 1 \end{bmatrix} = \omega^2 \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix} \quad (\omega^4 = \omega,\ \omega^3 = 1)$$

$\Rightarrow y$ is an eigenvector w/ eigenvalue ω^2.

6. What is $\frac{1}{\sqrt{3}}e^{i\pi/2} + e^{3i\pi}$?

A) $\frac{1}{\sqrt{3}}e^{2i\pi/3}$ B) $\sqrt{\frac{4}{3}}e^{-i\pi/6}$ C) $\frac{2}{3}e^{-7i\pi/6}$

D) $\sqrt{\frac{2}{3}}$ E) $\frac{2}{\sqrt{3}}e^{5i\pi/6}$ (circled)

$$\tfrac{1}{\sqrt{3}} e^{\frac{\pi}{2}i} + e^{3\pi i} = \tfrac{1}{\sqrt{3}}\left(\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{0} + i\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1}\right) + \underbrace{\cos(3\pi)}_{-1} + i\underbrace{\sin(3\pi)}_{0}$$

$$= \tfrac{1}{\sqrt{3}}i - 1 = -1 + \tfrac{1}{\sqrt{3}}i.$$

$$r = \sqrt{1^2 + \left(\tfrac{1}{\sqrt{3}}\right)^2} = \sqrt{1 + \tfrac{1}{3}} = \sqrt{\tfrac{4}{3}} = \tfrac{2}{\sqrt{3}}.$$

$$-1 + \tfrac{1}{\sqrt{3}}i = \tfrac{2}{\sqrt{3}}\left(\underbrace{-\tfrac{\sqrt{3}}{2}}_{\cos\theta} + \underbrace{\tfrac{1}{2}}_{\sin\theta}i\right) = \boxed{\tfrac{2}{\sqrt{3}}e^{\frac{5\pi}{6}i}}.$$

$\pi - \frac{\pi}{6} = \frac{5\pi}{6}.$

7. What is $(1+i)^5$?

A) $-\dfrac{1}{2\sqrt{8}}$ B) $-4(1+i)$ C) $\sqrt{32}i - \sqrt{32}$

D) $\sqrt{2}^5 i$ E) $\sqrt{8i} - \sqrt{8i}$

$r = |1+i| = \sqrt{1^2+1^2} = \sqrt{2}.$

$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{\frac{\pi}{4}i}.$

$(1+i)^5 = \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^5 = 2^{5/2}e^{\frac{5\pi}{4}i} = 2^{5/2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$= \sqrt{32}\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right] = \frac{\sqrt{32}}{\sqrt{2}}[-1-i] = -4[1+i].$

8. In $\mathbb{R}^3$, what is the distance between the point $(3,0,1)$ and the plane $x-y+z-6=0$?

A) 0 B) $\dfrac{4}{\sqrt{3}}$ C) $\dfrac{4}{3}$ D) $\dfrac{2}{3}$ E) $\dfrac{2}{\sqrt{3}}$

$P(3,0,1).$

$Q(6,0,0)$ is a point on the plane.

$\vec{PQ} = (3,0,-1).$

Plane has normal vector $\vec{n} = (1,-1,1).$

$d = \text{Proj}_{\vec{n}}\vec{PQ} = \dfrac{\begin{pmatrix}3\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}}{3}(1,-1,1) = \frac{2}{3}(1,-1,1).$

$\|d\| = \frac{2}{3}\sqrt{1^2+1^2+1^2} = \frac{2\sqrt{3}}{3} = \boxed{\frac{2}{\sqrt{3}}}.$

9. Which of the following are the parametric equations of the line in $\mathbb{R}^4$ that goes through the points $(1, 2, 0, 0)$ and $(3, 1, 0, 1)$?

A) $x_1 = 1 + 2t_1$, $x_2 = 1 - t_2$, $x_3 = 0$, $x_4 = t_2$

B) $x_1 = 2t$, $x_2 = 2 - t$, $x_3 = 0$, $x_4 = t$

C) $x_1 = 1 + 2t_1$, $x_2 = 2 - t_2$, $x_3 = t_1$, $x_4 = t_2$

D) $x_1 = 1 + 2t$, $x_2 = 2 - t$, $x_3 = 0$, $x_4 = 1 + t$

E) $x_1 = 1 + 2t$, $x_2 = 2 - t$, $x_3 = 0$, $x_4 = t$

$P(1,2,0,0)$

$Q(3,1,0,1)$.

$\overrightarrow{PQ} = (2,-1,0,1)$.

$l = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$.

$x_1 = 2t+1$
$x_2 = -t+2$
$x_3 = 0$
$x_4 = t$.

10. Let **u** and **v** be two unit vectors in $\mathbb{R}^3$ such that the projection of **v** along **u** is $\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{1}{5}(2, 0, -4)$. What is the absolute value of the cosine of the angle between the vectors **u** and **v**?

A) 0 B) 1 C) $\frac{4}{5}$ D) $\frac{2\sqrt{5}}{5}$ E) $\frac{4\sqrt{5}}{5}$

$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|} = \langle u, v\rangle$

$\frac{1}{5}(2,0,-4) = \text{proj}_u v = \frac{\langle u, v\rangle}{\|v\|^2} v = \langle u, v\rangle v = \cos\theta\, v$.

$\|\frac{1}{5}(2,0,-4)\| = |\cos\theta|\,\|v\| \Rightarrow |\cos\theta| = \frac{1}{5}\sqrt{20} = \frac{2}{5}\sqrt{5}$.

$u = (u_1, \dots, u_n)$.

11. Let $\mathbf{u} \in \mathbb{R}^n$. Which of the following statements must be true?

i) $\|\mathbf{u}\| \geq 0$, and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

ii) $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$, for every $\mathbf{v} \in \mathbb{R}^n$.

iii) $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, for every $\mathbf{v} \in \mathbb{R}^n$ which is orthogonal to $\mathbf{u}$.

(i) $u=0 \Rightarrow \|u\| = 0 \geq 0$. ✓ $\|u\|=0 \Rightarrow \sqrt{u_1^2 + \dots + u_n^2} = 0 \Rightarrow u_i = 0 \ \forall i \Rightarrow u = 0$. ✓

A) ii) and iii) only B) i) and iii) only C) i) only

D) ii) only E) i), ii) and iii)

(ii) $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle \Rightarrow \|u+v\| = \sqrt{\langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle}$
$\neq \sqrt{\langle u,u\rangle} + \sqrt{\langle v,v\rangle} = \|u\| + \|v\|$ ✗

(iii) $\|u+v\|^2 = \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle = \|u\|^2 + \|v\|^2$.
$\langle u,v\rangle = 0$ if $u \perp v$

12. What is the equation of the plane in $\mathbb{R}^3$ that goes through the origin and that is parallel to the vectors $(1,1,1)$ and $(0,1,2)$?

A) $x - 2y + z = 0$ B) $x + y + z = 0$ C) $-2x + y + z = 0$

D) $x + y - 2z = 0$ E) $x - 2y + z + 1 = 0$

vector normal to plane

$$\vec{n} = u \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$= i - 2j + k = (1,-2,1)$.

$x - 2y + z = 0$ is the eq'n of plane & goes through $(0,0,0)$ since $0 - 2(0) + 0 = 0$. ✓

13. In $\mathbb{R}^3$, what is the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ of $\mathbf{u} = (1,0,1)$, $\mathbf{v} = (0,0,3)$ and $\mathbf{w} = (2,2,2)$?

A) -2 B) -6 C) 0 D) 3 E) 2

$u \cdot (v \times w) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 2 & 2 & 2 \end{vmatrix} = -3 \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = -3 \cdot 2 = -6.$

14. Consider the subset of skew symmetric matrices in the vector space of real 2x2 matrices. (Remember, saying a matrix A is skew symmetric means that $A^T = -A$)
Which of the following is a true statement:

(A) The subset is closed under addition and scalar multiplication, and is a subspace.

(B) The subset is closed under addition and scalar multiplication, and is **not** a subspace.

(C) The subset is closed under addition, but **not** scalar multiplication, and is **not** a subspace.

(D) The subset is closed under **neither** addition **nor** multiplication, and is **not** a subspace.

(E) The subset is closed under scalar multiplication, but **not** addition, and is **not** a subspace.

or can prove for any size matrix: A & B skew-sym $\Rightarrow A = -A^T$, $B = -A^T$. $(A+B)^T = A^T + B^T = -A - B = -(A+B)$. ✓ $(kA)^T = kA^T = -kA$. ✓

$X = \{ A \in M_2(\mathbb{R}) \mid A = -A^T \}$. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $-A^T = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$

$A = -A^T \Rightarrow$ $a = -a \Rightarrow 2a = 0 \Rightarrow a = 0$, $d = -d \Rightarrow 2d = 0 \Rightarrow d = 0$. $b = -c$. $X = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \mid b \in \mathbb{R} \right\}$.

"+": Let $A, B \in X$. So $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ for some $a, b \in \mathbb{R}$.

$A + B = \begin{bmatrix} 0 & a+b \\ -a-b & 0 \end{bmatrix}$. $(A+B)^T = \begin{bmatrix} 0 & -a-b \\ a+b & 0 \end{bmatrix} = -(A+B) \Rightarrow A+B$ skew-sym. $\Rightarrow A+B \in X$. ✓

"α": $\alpha A = \alpha \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a \\ -\alpha a & 0 \end{bmatrix} \in X$. ✓ $(b = \alpha a)$.

15. Let **v** be the set of real ordered triples, (a, b, c), with the usual addition rules, and a new scalar multiplication operation given by:

$$k(a, b, c) = (0, kb, 0)$$

And consider the Real Vector Space Axioms, where given $\mathbf{v} \in V$ and any real scalars, l and k:

$v = (a, b, c)$.

- #7: $l(k(\mathbf{v})) = (lk)(\mathbf{v})$
- #9: $(k + l)\mathbf{v} = k\mathbf{v} + l\mathbf{v}$
- #10: $1\mathbf{v} = \mathbf{v}$

Which of the following statements is true?

(A) #9 holds, #7 and #10 do not.

(B) All three of the given axioms hold.

(C) #7 holds, #9 and #10 do not.

(D) None of the given axioms hold.

(E) #7 and #9 hold, #10 does not.

#7: $l(k(v)) = l(0, kb, 0) = (0, lkb, 0) = (lk)(v)$. ✓

#9: $(k+l)v = (0, (k+l)b, 0)$. same ✓

$kv + lv = (0, kb, 0) + (0, lb, 0) = (0, (k+l)b, 0)$.

#10: $1v = (0, b, 0) \neq v = (a, b, c)$. X

16. Given the polynomial $\mathbf{q} = x^2 - 3$, we can express it as a linear combination of:

$$\mathbf{p}_1 = 1 - x + x^2,\ \mathbf{p}_2 = x + 1,\ \mathbf{p}_3 = x - 1$$

What will be the coefficient of $\mathbf{p}_3$?

A) 2 B) 3/2 C) 1 D) 5/2 E) 0

$a(1 - x + x^2) + b(x+1) + c(x-1) = x^2 - 3$

$\Rightarrow x^2(a) + x(-a + b + c) + (a + b - c) = x^2 - 3$

$\Rightarrow a = 1$ & $-a + b + c = 0$ & $a + b - c = -3$

$\Rightarrow b + c = 1$ $\Rightarrow b - c = -4$

$\Rightarrow b = 1 - c$ $\Rightarrow 1 - c - c = -4$

$\Rightarrow -2c = -5 \Rightarrow \boxed{c = \frac{5}{2}} \Rightarrow b = 1 - \frac{5}{2} = -\frac{3}{2}.$

17. Which of the following statements are **always** true about the set of real vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

(i) $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{0}\}$ ✓
(That is, any element of one set is an element of the other)

(ii) If $\mathbf{w} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ then $\mathbf{w} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ ✓ $\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3\}$.

(iii) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are elements of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

↳ not necessarily. what if $u_2 = -u_1$ & $u_3 = 2u_1$?

A) (i) and (ii) B) (i) and (iii) C) (ii)

D) All the statements are always false. E) All the statements are always true.

e.g. $W = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \mathbb{R}^3.$

$\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix}-1\\0\\0\end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix}2\\0\\0\end{bmatrix}}_{u_3} \in W$

but $\text{span}\{u_1, u_2, u_3\}$ = a line... not $\mathbb{R}^3$.

18. In Matlab, we can assign a vector, v, as: `[1 2 3]`
Which of the following Matlab commands creates the matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

A) `offset(v,1)` B) `tril(v,4)` C) `lower(v)`

D) `diag(v,-1)` E) `angle(lower,v,1)`

19. A new association has been formed to officially certify and register eccentrics. There are many people preparing for their certification. At the beginning of each month, 1/4 of those who have been preparing go in for written and oral examinations. They do their exams for a month, and at the end, 40 of every 50 people who have been doing exams earn their certificate as a registered eccentric. The rest go back into preparing to try again. A certified eccentric keeps their certification forever.

If we construct a state vector (p_t, x_t, c_t) for the proportions of a specific group of candidates that are preparing (p_t), being examined (x_t), or who are certified (c_t), what is the monthly transition matrix associated with this process?

A) $\begin{bmatrix} 0.25 & 40 & 1 \\ 0 & 0 & 0 \\ 0.75 & 10 & 0 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 1/5 & 1 \\ 1/4 & 1 & 0 \\ 0 & 4/5 & 1 \end{bmatrix}$ C) $\begin{bmatrix} 0.25 & 0.2 & 0 \\ 0 & 0.8 & 0 \\ 0.75 & 0 & 1 \end{bmatrix}$

D) $\begin{bmatrix} 40 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 10 & 0 & 1/2 \end{bmatrix}$ (E) $\begin{bmatrix} 0.75 & 0.2 & 0 \\ 0.25 & 0 & 0 \\ 0 & 0.8 & 1 \end{bmatrix}$

$p_t \quad x_t \quad c_t$

$$M = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 3/4 & 1/5 & 0 \\ 1/4 & 0 & 0 \\ 0 & 4/8 & 1 \end{bmatrix}.$$

$P_{11} = 0.75$ (prepare → prepare).
$P_{21} = 0.25$ (prepare → exam).
$P_{31} = 0$.

$P_{12} = 10/50 = 1/5$
$P_{22} = 0$
$P_{32} = 40/50 = 4/5$

$P_{13} = 0$
$P_{23} = 0$
$P_{33} = 1$.

20. A population of mice can be described with a state vector (y_t, m_t, e_t) where y_t is the number of young mice, m_t is the number of middle-aged mice, and e_t is the number of elderly mice. The evolution of the population over time can be modeled as a dynamical system with a yearly transition matrix, A. Suppose the transition matrix, A, has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -2$, and $\lambda_3 = 0$, with corresponding eigenvectors $\mathbf{x_1} = (5, 2, 1)$, $\mathbf{x_2} = (5, -2, 1)$, and the third eigenvector $\mathbf{x_3}$ unspecified. The initial population is given by initial state vector $100\mathbf{x_1} + 10\mathbf{x_2} + 20\mathbf{x_3}$. How many middle-aged mice will there be 2 years later?

A diagonalizable b/c 3 distinct eigenvalues.

A) 1020 B) 940 C) 880 D) 720 E) 540

$$x_2 = A^2 x_0 = PD^2P^{-1}x_0 = \begin{bmatrix} 5 & 5 & * \\ 2 & -2 & * \\ 1 & 1 & * \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{P^{-1}P}_{I} \underbrace{\begin{bmatrix} 100 \\ 10 \\ 20 \end{bmatrix}}_{x_0}$$

$$= \begin{bmatrix} 5 & 5 & * \\ 2 & -2 & * \\ 1 & 1 & * \end{bmatrix}\begin{bmatrix} 400 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 2200 \\ 720 \\ 440 \end{bmatrix} \begin{matrix} y_t \\ m_t \\ e_t \end{matrix}$$

OR: can do it w/out diagonalizing: $A^2 x_0 = A^2(100x_1 + 10x_2 + 20x_3)$

$= 2^2(100)x_1 + (-2)^2(10)x_2 + 0^2(20)x_3 = 400x_1 + 40x_2$

$$= 400\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + 40\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2200 \\ 720 \\ 440 \end{bmatrix}.$$

$x_0 = 100\vec{x}_1 + 10\vec{x}_2 + 20\vec{x}_3$

$$= 100\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + 20\begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 5 & 5 & * \\ 2 & -2 & * \\ 1 & 1 & * \end{bmatrix}\begin{bmatrix} 100 \\ 10 \\ 20 \end{bmatrix} = P\begin{bmatrix} 100 \\ 10 \\ 20 \end{bmatrix}$$

END OF TEST QUESTIONS

Math 1B03

1st Sample Test #2

Name: DeDieu Lauren

(Last Name) (First Name)

Student Number:____________ **Tutorial Number:**____________

This test consists of 20 multiple choice questions worth 1 mark each (no part marks), and 1 question worth 1 mark (no part marks) on proper computer card filling. All questions must be answered on the COMPUTER CARD with an HB PENCIL. Marks will not be deducted for wrong answers (i.e., there is no penalty for guessing). You are responsible for ensuring that your copy of the test is complete. Bring any discrepancy to the attention of the invigilator. Calculators are NOT allowed.

1. Determine which of the following matrices is a regular stochastic matrix, and then find the steady-state vector for the associated Markov Chain.

$$A = \begin{bmatrix} \frac{1}{5} & 0 \\ \frac{4}{5} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

(a) $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ **(b)** $\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$ **(c)** $\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$ **(d)** $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ **(e)** $\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$ (see paper)

2. After exposure to certain live pathogens, the body develops long-term immunity. The evolution over time of the associated disease can be modeled as a dynamical system whose state vector at time t consists of the number of people who have not been exposed and are therefore susceptible, the number who are currently sick with the disease, and the number who have recovered and are now immune. Suppose that the associated 3×3 yearly transition matrix A has eigenvalues $\lambda = 1, \frac{1}{2}, 0$, and that the eigenvectors corresponding to the first two eigenvalues are $\mathbf{x}_1 = (60, 20, 30)$ and $\mathbf{x}_2 = (-60, -30, 90)$, respectively. The initial state vector for the population is given by

$$\mathbf{v}_0 = 500\mathbf{x}_1 + 200\mathbf{x}_2 + 100\mathbf{x}_3$$

where the third eigenvector $\mathbf{x}_3$ is not given here. How many people will be sick with the disease 2 years later?

(a) 15450 **(b)** 27000 **(c)** 8500 **(d)** 9700 **(e)** 4000 (see paper)

3. Find the equation of the plane passing through $A(2,1,3), B(3,-1,5)$, and $C(1,2,-3)$.

(a) $4x - 2z - 2 = 0$ **(b)** $10x + 4y - z - 21 = 0$
(c) $6x - 3z - 3 = 0$ **(d)** $8x + y - 3z - 8 = 0$
(e) $6x - 2y - 5z + 5 = 0$ (see paper)

$\overrightarrow{PQ} = (0, 0, 1)$ ← slope (direction)

So, $l = \begin{pmatrix}0\\0\\1\end{pmatrix} t + \begin{pmatrix}3\\-1\\4\end{pmatrix}$

$x = 3$
$y = -1$
$z = t + 4$

4. Find the parametric equations of the line passing through the points $P(3, -1, 4)$ and $Q(3, -1, 5)$.

(a)	(b)	(c)	(d)	(e)
$x = 3t$	$x = 3$	$x = 3 + t$	$x = t$	$x = 0$
$y = -t$	$y = -1$	$y = -1 + t$	$y = t$	$y = 0$
$z = 1 + 4t$	$z = 4 + t$	$z = 4 + t$	$z = 1 + 4t$	$z = t$

5. Find the volume of the parallelepiped determined by **w**, **u**, and **v** when: $\mathbf{w} = (2, 1, 1)$, $\mathbf{v} = (1, 0, 2)$, and $\mathbf{u} = (2, 1, -1)$.
(a) 1 (b) 2 (c) 3 (d) 4 (e) 5 (See Paper)

6. Find the shortest distance between the following pairs of parallel lines.
$(x, y, z) = (2, -1, 3) + t(1, -1, 4)$
$(x, y, z) = (1, 0, 1) + t(1, -1, 4)$
(a) $\frac{5}{9}$ (b) $\frac{2}{9}$ (c) 1 (d) $\frac{2}{3}$ (e) $\frac{1}{\sqrt{2}}$ (See Paper)

7. A parallelogram has sides AB, BC, CD, and DA. Given $A(1, -1, 2), C(2, 1, 0)$, and the midpoint $M(2, 0, -3)$ of AB, find $\overrightarrow{BD}$.
(a) $(3, 1, -8)$ (b) $(-1, 0, 8)$ (c) $(2, 2, -10)$ (d) $(-3, -2, 18)$ (e) $(1, -1, 2)$ (See Paper)

8. Consider the following statements regarding vectors in R^3.
(i) If $\|\mathbf{u}\| = \|\mathbf{v}\|$ then $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{u} - \mathbf{v}$
(ii) If $\mathbf{v}$ is orthogonal to $\mathbf{w}_1$ and $\mathbf{w}_2$, then $\mathbf{v}$ is orthogonal to $\mathbf{u} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2$ for all scalars k_1 and k_2.
(iii) $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$ is an independent set.

Which of the above statements are always true?
(a) (i) only
(b) (ii) only
(c) (i) and (ii)
(d) (iii) only
(e) (ii) and (iii)

(i) $(u+v)\cdot(u-v) = u\cdot(u-v) + v\cdot(u-v) = u\cdot u - u\cdot v + v\cdot u - v\cdot v = \|u\|^2 - \|v\|^2 = \|u\|^2 - \|u\|^2 = 0 \Rightarrow$ orthogonal. ✓
Since $\|u\| = \|v\| \Rightarrow \|u\|^2 = \|v\|^2$

(ii) Given $v\cdot w_1 = 0$ & $v\cdot w_2 = 0$.
$v\cdot u = v\cdot(k_1w_1 + k_2w_2) = v\cdot(k_1w_1) + v\cdot(k_2w_2) = k_1(v\cdot w_1) + k_2(v\cdot w_2) = 0$
$\Rightarrow v$ & u orthogonal. ✓

9. Assume **u** and **v** are nonzero vectors that are not parallel. Let $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$. Find a simplified expression for the cosine of the angle between **u** and **w**.
(a) $\frac{\|\mathbf{u}\|[(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|]}{\|\mathbf{w}\|}$ (b) $\frac{(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|\|\mathbf{u}\|^2}{\|\mathbf{w}\|}$ (c) $\frac{2(\mathbf{u}\cdot\mathbf{v})}{\|\mathbf{w}\|}$ (d) $\frac{(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|\|\mathbf{u}\|}{\|\mathbf{w}\|}$ (e) $\frac{\|\mathbf{v}\|\|\mathbf{u}\|}{\|\mathbf{w}\|}$

(iii) It is an independent set if $a_1(u-v) + a_2(v-w) + a_3(w-u) = 0$ has only the trivial solution (i.e. $a_1 = a_2 = a_3 = 0$). But, rearranging we have:
$u(a_1 - a_3) + v(a_2 - a_1) + w(a_3 - a_2) = 0$
$\Rightarrow a_1 = a_3,\ a_1 = a_2,\ a_2 = a_3$.

So, try $a_1 = a_2 = a_3 = 1$:
$(u-v) + (v-w) + (w-u) = 0$
$\Rightarrow (1,1,1)$ is a solution
$\Rightarrow$ does not only have the trivial solution $\Rightarrow$ Not an independent set. ✗

We know $\cos\theta = \frac{u\cdot w}{\|u\|\|w\|} \Rightarrow$
$\cos\theta = \frac{u\cdot(\|u\|v + \|v\|u)}{\|u\|\|w\|}$
$= \frac{\|u\|u\cdot v + \|v\|u\cdot u}{\|u\|\|w\|}$
$= \frac{\|u\|u\cdot v + \|v\|\|u\|^2}{\|u\|\|w\|}$
$= \frac{u\cdot v + \|v\|\|u\|}{\|w\|}$.

10. Let V be the set of all ordered pairs of real numbers $\mathbf{u} = (u_1, u_2)$ with $u_1 > 0$, with the usual scalar multiplication, and consider the following operation on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$$\mathbf{u} + \mathbf{v} = (u_1 v_1, u_2 + v_2)$$

If this set satisfies Axiom 4 of a vector space (the existence of a zero vector), what would be the zero vector?

(a) $(1, 0)$ **(b)** $(0, 0)$ **(c)** $(0, 1)$ **(d)** $(1, 1)$ **(e)** $(0, -1)$ (see paper)

11. Let V be the set of all ordered pairs of real numbers $\mathbf{u} = (u_1, u_2)$ with $u_1 > 0$, with the usual scalar multiplication, and consider the following operation on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$$\mathbf{u} + \mathbf{v} = (u_1 v_1, u_2 + v_2)$$

If this set satisfies Axiom 5 of a vector space (the existence of the negative of a vector), what would be the value of $-\mathbf{u}$?

(a) $(\frac{1}{u_2}, -u_1)$ **(b)** $(-u_1, -u_2)$ **(c)** $(u_1, -u_2)$ **(d)** $(-u_1, u_2)$ **(e)** $(\frac{1}{u_1}, -u_2)$ (see paper)

For Questions 12-14, determine which of the following answers is correct for the given subset W of $\mathbb{R}^3$.

(a) W is a subspace
(b) W is closed under addition, but not closed under scalar multiplication
(c) W is closed under scalar multiplication, but not closed under addition
(d) W is not closed under scalar multiplication, and not closed under addition

12. $W =$ all vectors of the form $(a, b, 1)$

(a) **(b)** **(c)** **(d)**

Let $u, v \in W \Rightarrow u = (a, b, 1)$ & $v = (c, d, 1)$ for some $a, b, c, d \in \mathbb{R}$. Then, $u + v = (a+c, b+d, 2) \notin W$. ✗ $\alpha u = (\alpha a, \alpha b, \alpha) \notin W$. ✗ So not closed under addition or scalar multiplication.

13. $W =$ all vectors of the form (a, b, c) where $a - 2c = 0$

(a) **(b)** **(c)** **(d)**

Let $u, v \in W \Rightarrow u = (a, b, c)$ s.t. $a - 2c = 0$ & $v = (d, e, f)$ s.t. $d - 2f = 0$. $u + v = (a+d, b+e, c+f)$. $a + d - 2(c+f) = (a - 2c) + (d - 2f) = 0 \Rightarrow u + v \in W$. ✓ $\alpha u = (\alpha a, \alpha b, \alpha c)$. $\alpha a - 2\alpha c = \alpha(a - 2c) = \alpha(0) = 0 \Rightarrow \alpha u \in W$. ✓

14. $W =$ all vectors of the form (a, b, c) where $c \geq 0$

(a) **(b)** **(c)** **(d)**

Let $u, v \in W \Rightarrow u = (a, b, c)$ s.t. $c \geq 0$ & $v = (d, e, f)$ s.t. $f \geq 0$. $u + v = (a+d, b+e, c+f)$. We know $c \geq 0$ & $f \geq 0 \Rightarrow c + f \geq 0 \Rightarrow u + v \in W$. ✓ $\alpha u = (\alpha a, \alpha b, \alpha c)$. If $\alpha = -1$, then $-1 * c \leq 0$ since $c \geq 0 \Rightarrow \alpha u \notin W$. ✗

15. Let $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$. Which of the following vectors are in span$\{\mathbf{u}, \mathbf{v}\}$?

(i) $(1, -1, 2)$
(ii) $(1, 1, 1)$
(iii) $(5, 3, 7)$

(a) all of them **(b)** (ii) only **(c)** (iii) only **(d)** (i) and (iii) only **(e)** (ii) and (iii) only

$(1, -1, 2) \in \text{span}\{u, v\}$ if $\exists\, \alpha_1, \alpha_2 \in \mathbb{R}$ s.t. $\alpha_1 u + \alpha_2 v = (1, -1, 2)$;

i.e. $\alpha_1 \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \alpha_2 \\ -2\alpha_1 + 3\alpha_2 \\ 2\alpha_1 - \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \Leftrightarrow \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(see paper)

16. If $\{\mathbf{v}, \mathbf{w}\}$ is independent, find conditions on the scalars k_1 and k_2 so that the set $\{k_1\mathbf{v} + \mathbf{w}, \mathbf{v} + k_2\mathbf{w}\}$ is also independent.

(a) $k_1 + k_2 = 1$ **(b)** $k_1 + k_2 \neq 1$ **(c)** $k_1 \neq k_2$ **(d)** $k_1k_2 = 1$ **(e)** $k_1k_2 \neq 1$

(see paper)

17. Find the reduced row echelon form of the following matrix.

$$\begin{bmatrix} -1 & -i & 1 \\ -i & 1 & i \\ 1 & i & -1 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 & i & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

18. Find $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^{60}$.

(a) 1 **(b)** -1 **(c)** $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ **(d)** $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ **(e)** $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(see paper)

19. Recall that $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote, respectively, the real and imaginary parts of the complex number z. Consider the following statements.

(i) $\mathrm{Re}(iz) = \mathrm{Im}(z)$

(ii) $z - \overline{z} = 2i\,\mathrm{Re}(z)$

Which of the above statements is always true?

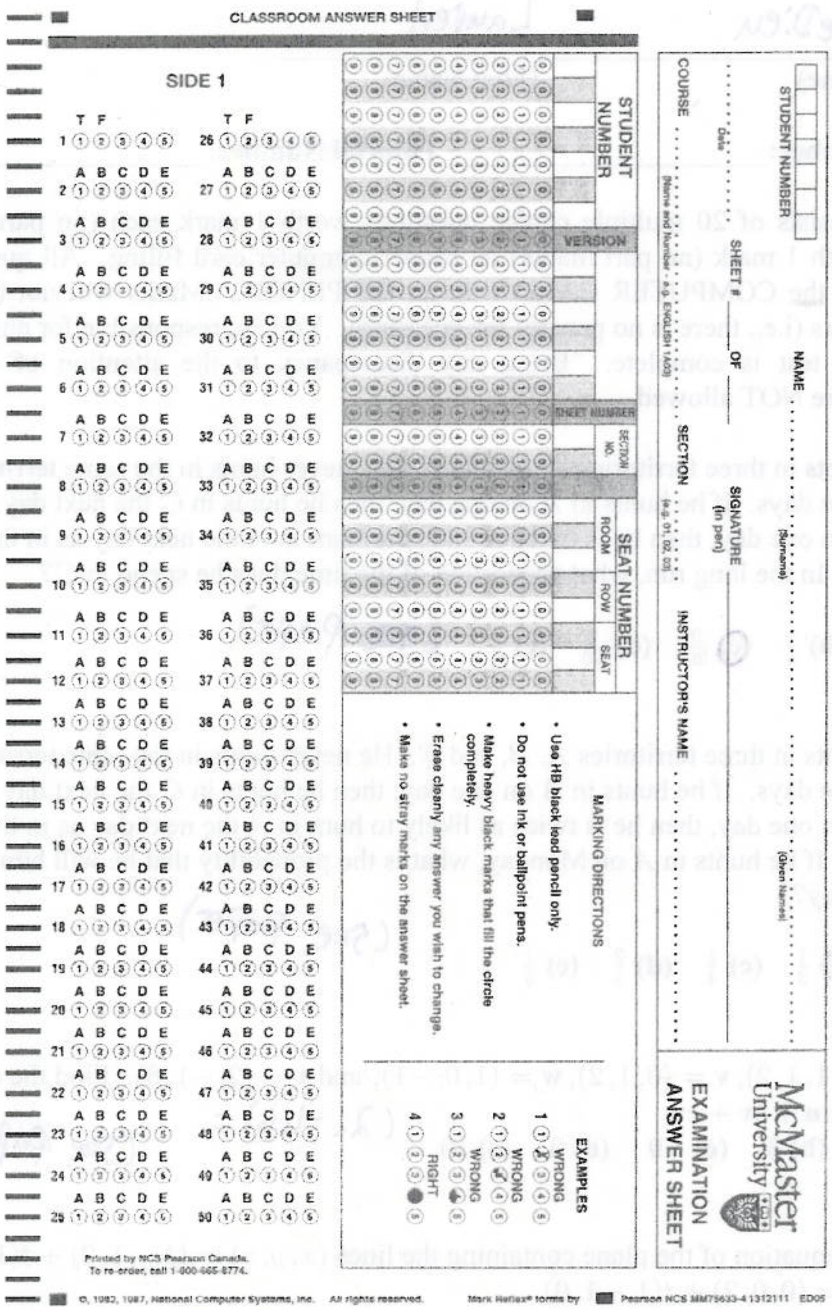
(a) (i) only **(b)** (ii) only **(c)** (i) and (ii) **(d)** neither

20. In Matlab, what command could be used to produce the following matrix?

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

(a) `>>diag(1,1,3,2)` **(b)** `>>block(eye(2),3,2)` **(c)** `>>ones(3,2)`
(d) `>>repmat(eye(2),3,2)` **(e)** `>>eye(3,2)`

21. Correctly fill out the bubbles corresponding to your student number and the version number of your test in the correct places on the computer card. (Use the below computer card for this sample test.)



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VERSION

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SECTION NO.

SEAT NUMBER

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- Use HB black lead pencil only.
- Do not use ink or ballpoint pens.
- Make heavy black marks that fill the circle completely.
- Erase cleanly any answer you wish to change.
- Make no stray marks on the answer sheet.

EXAMPLES

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2 WRONG

3 WRONG

4 RIGHT

McMaster University

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Math 1B03

2nd Sample Test #2

Name: DeDieu Lauren
(Last Name) (First Name)

Student Number:__________ **Tutorial Number:**__________

This test consists of 20 multiple choice questions worth 1 mark each (no part marks), and 1 question worth 1 mark (no part marks) on proper computer card filling. All questions must be answered on the COMPUTER CARD with an HB PENCIL. Marks will not be deducted for wrong answers (i.e., there is no penalty for guessing). You are responsible for ensuring that your copy of the test is complete. Bring any discrepancy to the attention of the invigilator. Calculators are NOT allowed.

1. A fox hunts in three territories A, B, and C. He never hunts in the same territory on two successive days. If he hunts in A on one day, then he hunts in C the next day. If he hunts in B or C on one day, then he is twice as likely to hunt in A the next day as in the other territory. In the long run, what proportion of the time does he spend in C?

(a) $\frac{7}{20}$ **(b)** $\frac{1}{5}$ **(c)** $\frac{9}{20}$ **(d)** $\frac{3}{20}$ **(e)** $\frac{2}{5}$ (see paper)

2. A fox hunts in three territories A, B, and C. He never hunts in the same territory on two successive days. If he hunts in A on one day, then he hunts in C the next day. If he hunts in B or C on one day, then he is twice as likely to hunt in A the next day as in the other territory. If he hunts in A on Monday, what is the probability that he will hunt in B on Wednesday?

(see paper)

(a) $\frac{2}{5}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{2}{3}$ **(e)** $\frac{1}{5}$

3. Let $\mathbf{u} = (1, 1, 2)$, $\mathbf{v} = (0, 1, 2)$, $\mathbf{w} = (1, 0, -1)$, and $\mathbf{x} = (2, -1, 6)$. Find the number c such that $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$

(a) -8 **(b)** 8 **(c)** -9 **(d)** 9 **(e)** 10 (see paper)

4. Find the equation of the plane containing the lines $(x, y, z) = (1, -1, 2) + t(1, 1, 1)$ and $(x, y, z) = (0, 0, 2) + t(1, -1, 0)$.

(a) $x + y + z = 2$ **(b)** $x + y - 3z = -6$ **(c)** $x + y + 4z = 8$
(d) $x + y - 2z = -4$ **(e)** $x + y - z = -2$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = i\begin{vmatrix}1 & 1\\-1 & 0\end{vmatrix} - j\begin{vmatrix}1 & 1\\1 & 0\end{vmatrix} + k\begin{vmatrix}1 & 1\\1 & -1\end{vmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

← the normal to both lines

So, $x + y - 2z + d = 0$. Plugging $(1, -1, 2)$ in:
$1 - 1 - 4 + d = 0 \Rightarrow d = 4$.
$x + y - 2z = -4$.

5. Find the parametric equations of the line passing through $P(1, 0, -3)$ and parallel to the line with parametric equations $x = -1 + 2t$, $y = 2 - t$, and $z = 3 + 3t$.

(a) $x = 1 - t$, $y = t$, $z = -3 + t$

(b) $x = 1 + 2t$, $y = -t$, $z = -3 + 3t$

(c) $x = 1 - t$, $y = 2t$, $z = -3 + 3t$

(d) $x = 1$, $y = 2t$, $z = -3$

(e) $x = 1 - 2t$, $y = 2t$, $z = -3 + 6t$

$x = -1 + 2t$, $y = 2 - t$, $z = 3 + 3t$

$\ell = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} t$ ← direction

So, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} t$

$x = 1 + 2t$, $y = -t$, $z = -3 + 3t$.

6. Compute the projection of $\mathbf{u}$ onto $\mathbf{v}$. $\mathbf{u} = (5, 7, 1)$, $\mathbf{v} = (1, -1, 3)$

(a) $(\frac{54}{11}, 8, \frac{7}{11})$ (b) $\frac{1}{11}(1, -1, 3)$ (c) $\frac{1}{75}(5, 7, 1)$ (d) $\frac{1}{\sqrt{75}\sqrt{11}}$ (e) $(\frac{71}{75}, -\frac{82}{75}, \frac{224}{75})$

$\text{Proj}_v u = \frac{u \cdot v}{\|v\|^2} v = \frac{5 - 7 + 3}{1 + 1 + 9}(1, -1, 3) = \frac{1}{11}(1, -1, 3)$.

7. Let $P_1 = P_1(2, 1, -2)$ and $P_2 = P_2(1, -2, 0)$. Find the coordinates of the point P which is $\frac{1}{5}$ of the way from P_1 to P_2.

(a) $(\frac{3}{5}, -\frac{1}{5}, -\frac{2}{5})$ (b) $(-\frac{1}{5}, -\frac{3}{5}, \frac{2}{5})$ (c) $(\frac{6}{5}, -\frac{7}{5}, -\frac{2}{5})$

(d) $(-\frac{4}{5}, -\frac{12}{5}, \frac{8}{5})$ (e) $(\frac{9}{5}, \frac{2}{5}, -\frac{8}{5})$

$\overrightarrow{P_1P_2} = (-1, -3, 2)$.

$P = P_1 + \frac{1}{5}\overrightarrow{P_1P_2} = (2, 1, -2) + (-\frac{1}{5}, -\frac{3}{5}, \frac{2}{5}) = (\frac{9}{5}, \frac{2}{5}, -\frac{8}{5})$.

8. Find the area of the triangle with vertices $A(1, 1, -1)$, $B(2, 0, 1)$, and $C(1, -1, 3)$.

(a) $2\sqrt{5}$ (b) $\sqrt{5}$ (c) $\sqrt{3}$ (d) $2\sqrt{3}$ (e) $\sqrt{7}$

$\overrightarrow{AB} = (1, -1, 2)$. $\overrightarrow{AC} = (0, -2, 4)$.

$\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \|\frac{1}{2}\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 0 & -2 & 4 \end{vmatrix}\| = \|\frac{1}{2}[i\begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} + k\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}]\| = \|\frac{1}{2}\begin{pmatrix} 0 \\ -4 \\ -2 \end{pmatrix}\| = \|\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\| = \sqrt{(-2)^2 + (-1)^2} = \boxed{\sqrt{5}}$

9. Find the point on the plane $3x - y + 4z = 1$ closest to the point $P(2, 1, -3)$.

(a) $(\frac{8}{5}, \frac{11}{13}, \frac{7}{8})$ (b) $(\frac{7}{3}, -\frac{2}{3}, \frac{5}{3})$ (c) $(\frac{25}{11}, -\frac{23}{11}, \frac{9}{11})$

(d) $(-\frac{38}{13}, \frac{11}{13}, \frac{23}{13})$ (e) $(\frac{38}{13}, \frac{9}{13}, -\frac{23}{13})$

(see paper)

10. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$, and consider the following statements.

(i) $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{2}\|\mathbf{u} - \mathbf{v}\|^2$

(ii) $\mathbf{v} - \mathbf{w}$ and $(\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \times \mathbf{w}) + (\mathbf{u} \times \mathbf{w})$ are orthogonal.

(iii) If $\mathbf{u}$ is any vector then the projection of $\mathbf{u}$ on $\mathbf{v}$ equals the projection of $\mathbf{u} - \mathbf{v}$ on $\mathbf{v}$.

Which of the above statements are always true?

(a) (i) and (ii) only

(b) (i) and (iii) only

(c) (ii) and (iii) only

(d) (ii) only

(e) none of them

(i) $\frac{1}{2}\|u + v\|^2 - \frac{1}{2}\|u - v\|^2 = \frac{1}{2}(u+v)\cdot(u+v) - \frac{1}{2}(u-v)\cdot(u-v)$

$= \frac{1}{2}[u \cdot u + 2u \cdot v + v \cdot v] - \frac{1}{2}(u \cdot u - 2u \cdot v + v \cdot v)$

$= \frac{1}{2} u \cdot u + u \cdot v + \frac{1}{2} v \cdot v - \frac{1}{2} u \cdot u + u \cdot v - \frac{1}{2} v \cdot v = 2u \cdot v \neq u \cdot v$. ✗

(ii) $(v - w) \cdot [u \times v + v \times w + u \times w] = v \cdot u \times v + v \cdot v \times w + v \cdot u \times w - w \cdot u \times v - w \cdot v \times w - w \cdot u \times w$ ($w \cdot u \times v = v \cdot w \times u$)

$= v \cdot u \times w + v \cdot u \times w = 2v \cdot u \times w \neq 0$. ✗

(iii) $\text{Proj}_v u = \frac{u \cdot v}{\|v\|^2} v$. $\text{Proj}_v (u - v) = \frac{(u - v) \cdot v}{\|v\|^2} v = \frac{u \cdot v - v \cdot v}{\|v\|^2} v = \frac{u \cdot v}{\|v\|^2} v - \frac{\|v\|^2}{\|v\|^2} v = \text{Proj}_v u - v \neq \text{Proj}_v u$. ✗

11. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and

$$k\mathbf{u} = (ku_1 + 1, ku_2).$$

Recall axioms 7-9 of a vector space:

7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

9. $k(m\mathbf{u}) = (km)\mathbf{u}$

Which of these axioms are true?

(a) all of them **(b) 7.** only **(c)** none of them **(d) 7.** and **8.** only **(e) 9.** only

For Questions 12-14, determine which of the following answers is correct for the given subset W of $\mathbb{R}^3$.

(a) W is a subspace

(b) W is closed under addition, but not closed under scalar multiplication

(c) W is closed under scalar multiplication, but not closed under addition

(d) W is not closed under scalar multiplication, and not closed under addition

12. $W =$ all vectors of the form (a, b, c) where $a - 2c - 1 = 0$

(a) **(b)** **(c)** **(d)**

13. $W =$ all vectors of the form (a, b, c) where the product $ab \geq 0$.

(a) **(b)** **(c)** **(d)**

14. Let $\mathbf{y}$ be a given vector. Let $W =$ all vectors $\mathbf{x}$ such that $\mathbf{y} \cdot \mathbf{x} = 0$.

(a) **(b)** **(c)** **(d)**

15. The following set of vectors $\{(1, -1, 1, -1), (2, 0, 1, 0), (0, -2, 1, -2)\}$

(a) Spans $\mathbb{R}^4$ but is not independent.

(b) Is independent, but does not span $\mathbb{R}^4$

(c) Is independent and spans $\mathbb{R}^4$

(d) Is not independent, and does not span $\mathbb{R}^4$

16. Suppose that $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where each $\mathbf{v}_i$ is in $\mathbb{R}^3$. Consider the following statements.

(i) If $\mathbf{x}$ is in W then $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for some scalars c_1, c_2, c_3.
(ii) $3\mathbf{v}_1 - 2\mathbf{v}_2$ is in W.
(iii) W is a subspace of $\mathbb{R}^3$.

Which of the above statements are always true?

(a) (i) only
(b) (ii) only
(c) (ii) and (iii) only
(d) (i) and (ii) only
(e) (i), (ii), and (iii) (circled)

These all follow directly from the def'n of spanning.
(i) $x \in \text{span}\{v_1, v_2, v_3\} \Rightarrow x = c_1v_1 + c_2v_2 + c_3v$ for some $c_1, c_2, c_3 \in \mathbb{R}$ by def'n.
(ii) $\text{span}\{v_1, v_2, v_3\}$ consists of all possible linear combinations of $v_1, v_2, \& v_3$ i.e. $c_1v_1 + c_2v_2 + c_3v_3 \in W$ $\forall c_1, c_2, c_3 \in \mathbb{R}$, so if $c_1 = 3$, $c_2 = -2$, & $c_3 = 0$, this will of course be in $\text{span}\{v_1, v_2, v_3\}$.
(iii) the span of vectors is always a subspace! (it's nonempty, closed under addition, & closed under scalar multiplication).

17. Solve the following equation for the complex number z. $z(1+i) = \overline{z} + (3+2i)$

(a) $-3+5i$ **(b)** $8-3i$ (circled) **(c)** $3+5i$ **(d)** $5-3i$ **(e)** $3+8i$

Let $z = a+bi$.
$(a+bi)(1+i) = a - bi + 3 + 2i \Leftrightarrow a + ai + bi - b = a - bi + 3 + 2i \Leftrightarrow ai + 2bi - b = 3 + 2i \Leftrightarrow -b + (a+2b)i = 3 + 2i$
$\Rightarrow -b = 3 \Rightarrow b = -3$. And $\Rightarrow a + 2b = 2 \Rightarrow a + 2(-3) = 2 \Rightarrow a - 6 = 2 \Rightarrow a = 8 \Rightarrow z = 8 - 3i$.

18. Express the following complex number in polar form. $z = -\sqrt{3} + i$.

(a) $2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$ (circled) **(b)** $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ **(c)** $2(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$
(d) $2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ **(e)** $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$r = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$.
$z = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = r(\cos\theta + i\sin\theta)$.
$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$
So, $z = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$.

19. Find all complex numbers z such that

$$z^3 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

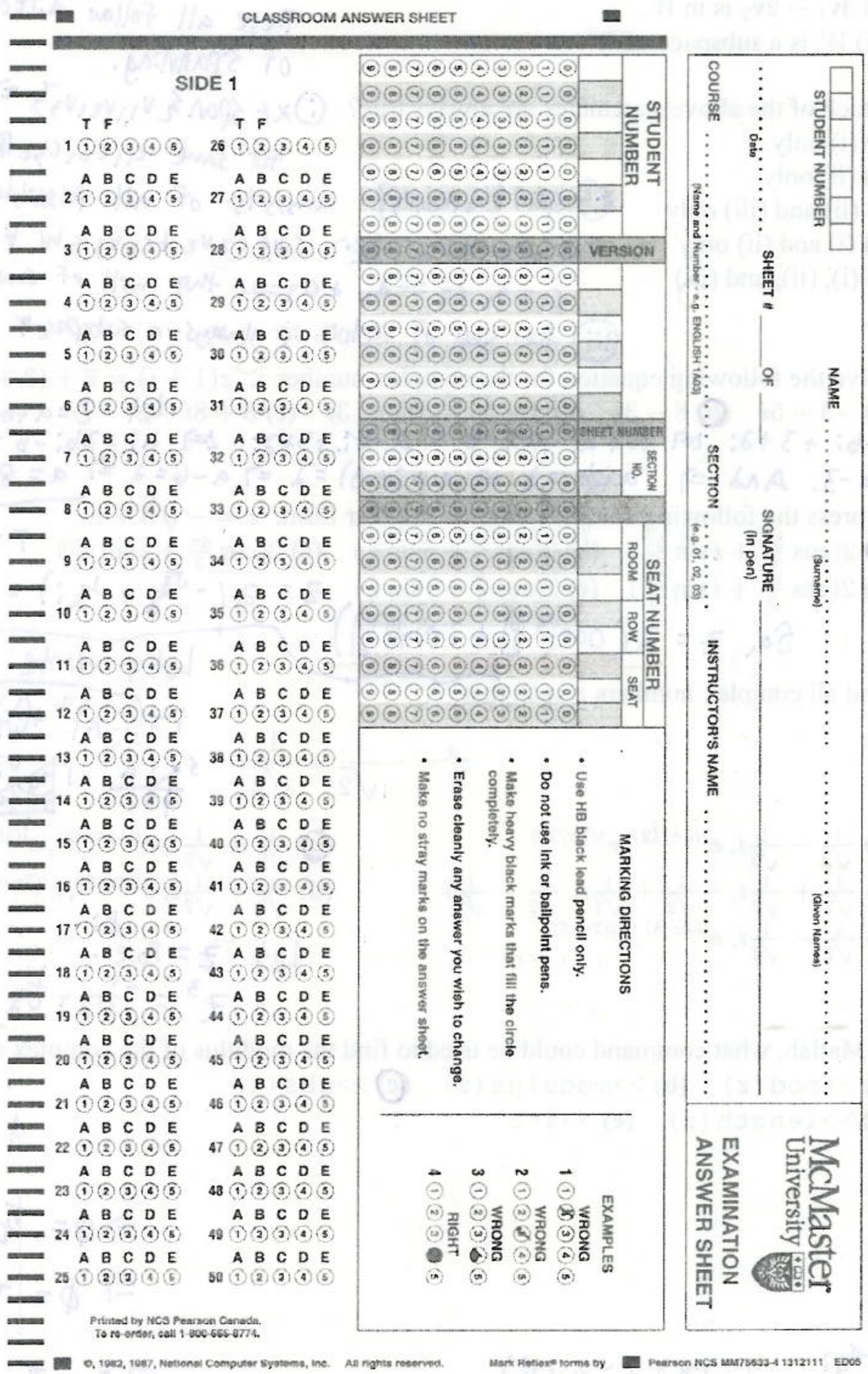
(a) $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,\ e^{i(5\pi/12)},\ e^{i(7\pi/12)}$
(b) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ e^{i(11\pi/12)},\ e^{i(19\pi/12)}$ (circled)
(c) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$
(d) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ e^{i(4\pi/5)},\ e^{i(7\pi/5)}$
(e) $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,\ e^{i(4\pi/5)},\ e^{i(7\pi/5)}$

Let's write $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ in polar form:
$r = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$.
$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.
So, $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = e^{\frac{3\pi}{4}i}$.
Let $z = r_0 e^{\phi i}$.
$z^3 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \Leftrightarrow r_0^3 e^{3\phi i} = e^{\frac{3\pi}{4}i}$
$\Rightarrow r_0^3 = 1 \Rightarrow r_0 = 1$
$\&\Rightarrow 3\phi = \frac{3\pi}{4} + 2k\pi$ for $k = 0, 1, 2$
$\Rightarrow \phi = \frac{\pi}{4} + \frac{2}{3}k\pi$
$\Rightarrow \phi = \frac{\pi}{4}$ or $\frac{\pi}{4} + \frac{2\pi}{3}$ or $\frac{\pi}{4} + \frac{4\pi}{3}$
$\Rightarrow \phi = \frac{\pi}{4}$ or $\frac{3\pi}{12} + \frac{8\pi}{12}$ or $\frac{3\pi}{12} + \frac{16\pi}{12}$
$\Rightarrow \phi = \frac{\pi}{4}$ or $\frac{11\pi}{12}$ or $\frac{19\pi}{12}$
$\Rightarrow z = e^{\frac{\pi}{4}i}$ or $e^{\frac{11\pi}{12}i}$ or $e^{\frac{19\pi}{12}i}$.
And $e^{\frac{\pi}{4}i} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

20. In Matlab, what command could be used to find the modulus of the complex number z?

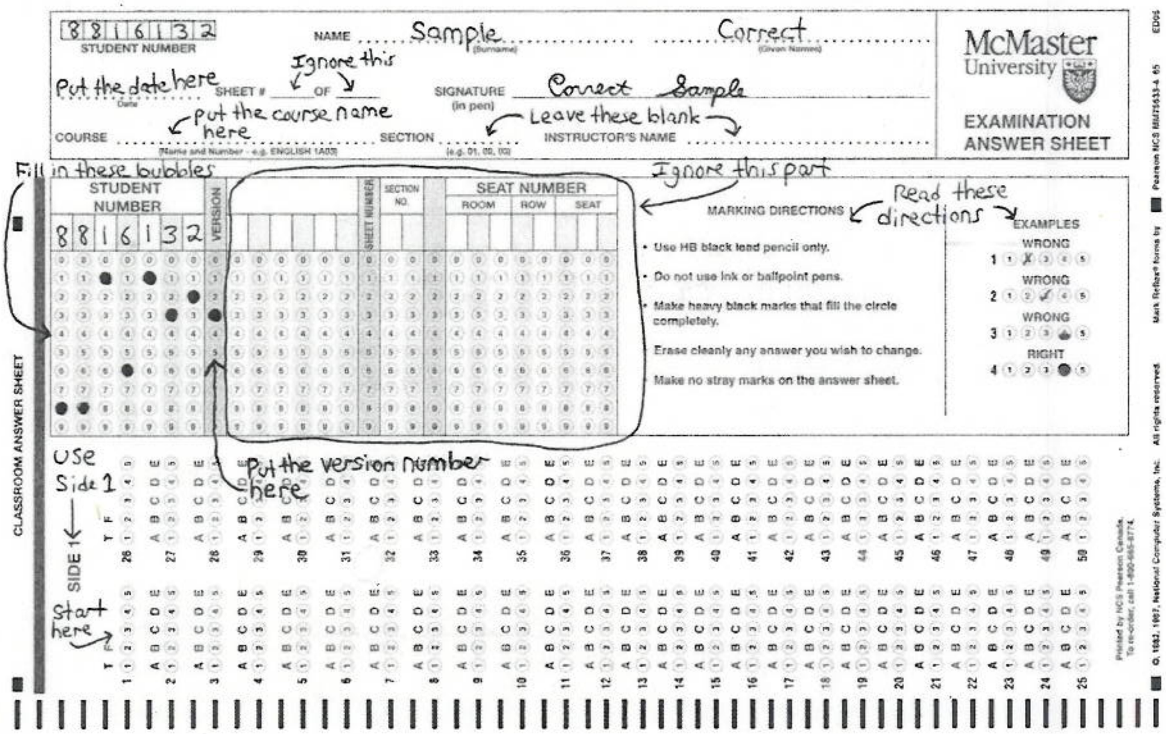
(a) `>>mod(z)` **(b)** `>>modulus(z)` **(c)** `>>abs(z)` (circled)
(d) `>>length(z)` **(e)** `>>z*z'`

21. Correctly fill out the bubbles corresponding to your student number and the version number of your test in the correct places on the computer card. (Use the below computer card for this sample test.)



Answers for 1st Sample Test #2

1. a **2.** c **3.** b **4.** b **5.** b **6.** d **7.** d **8.** c **9.** d **10.** a
11. e **12.** d **13.** a **14.** b **15.** e **16.** e **17.** a **18.** a **19.** d **20.** d
21.



NOTE: On the sample tests, a version number is not given. On the actual tests, it will say "Version X" at the top, where X is the version number that you will have to fill in on the computer card. The sample answer above assumes that the test says "Version 3" at the top. On the actual test you will have to fill in the bubble corresponding to the version number of YOUR test (which may or may not be Version 3). The sample above also assumes that your student number is 8816132. On the actual test, you will have to fill in the bubbles corresponding to YOUR student number (not 8816132).

Answers for 2nd Sample Test #2

1. c **2.** b **3.** a **4.** d **5.** b **6.** b **7.** e **8.** b **9.** e **10.** e
11. c **12.** d **13.** c **14.** a **15.** d **16.** e **17.** b **18.** a **19.** b **20.** c
21. see the answer to #21 on the first sample test above.

Answers for 1st Sample Test #2

1. a 2. c 3. b 4. b 5. b 6. d 7. d 8. c 9. d 10. a
11. e 12. d 13. a 14. b 15. c 16. c 17. a 18. a 19. d 20. d
21.

NOTE: On the sample tests, a version number is not given. On the actual tests, it will say "Version X" at the top, where X is the version number that you will have to fill in on the computer card. The sample answer above assumes that the test says "Version 3" at the top. On the actual test you will have to fill in the bubble corresponding to the version number of YOUR test (which may or may not be Version 3). The sample above also assumes that your student number is 88161322. On the actual test, you will have to fill in the bubbles corresponding to YOUR student number (not 88161322).

Answers for 2nd Sample Test #2

1. c 2. b 3. a 4. d 5. c 6. b 7. e 8. b 9. e 10. e
11. c 12. d 13. e 14. a 15. d 16. c 17. b 18. a 19. b 20. e
21. see the answer to 21. on the first sample test above.

Sample Test #1

1. Recall: • A **stochastic matrix** is a square matrix, each of whose columns is a probability vector.

• A stochastic matrix P is said to be regular if P, or some positive power of P, has all positive entries.

C's columns are probability vectors (i.e. a vector with nonnegative entries that add up to 1).

Also, $C^2 = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$

has all positive entries. ∴ C is a regular stochastic matrix.

Recall: If P is the transition matrix for a regular Markov chain, then there exists a unique **steady-state vector** $\vec{x}$ s.t. $P\vec{x} = \vec{x}$ (i.e. $\vec{x}$ is an eigenvector of P corresponding eigenvalue $\lambda = 1$).

$(C - \lambda I)\vec{x} = \vec{0}$.

$\left[\begin{array}{cc|c} 1/2 - 1 & 1 & 0 \\ 1/2 & -1 & 0 \end{array}\right]$ $\left[\begin{array}{cc|c} -1/2 & 1 & 0 \\ 1/2 & -1 & 0 \end{array}\right]$ $r_1 \leftarrow r_1 + r_2$ $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1/2 & -1 & 0 \end{array}\right]$

$\frac{1}{2}x - y = 0 \Rightarrow y = \frac{1}{2}x$, $x = t$. So $\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} t$ solves this system.

But, we need a probability vector, so need $t + \frac{1}{2}t = 1$ $\Rightarrow \frac{3}{2}t = 1 \Rightarrow t = \frac{2}{3}$ which gives us the vector $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.

transition matrix

initial state vector

2. Recall: $v_n = A^n v_0$

state vector for time n.

A has 3 distinct eigenvalues ⇒ A is diagonalizable ⇒ $A = PDP^{-1}$.

$$\text{So,}\quad A = \underbrace{\begin{bmatrix} 60 & -60 & * \\ 20 & -30 & * \\ 30 & 90 & * \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} P^{-1}.$$

$$v_n = \begin{bmatrix} \text{not exposed} \\ \text{sick} \\ \text{immune} \end{bmatrix}$$

In particular, we know $A^2 = PD^2P^{-1}$.

Now, we are interested in year 2:

$v_2 = A^2 v_0$.

We are given that $v_0 = 500\vec{x}_1 + 200\vec{x}_2 + 100\vec{x}_3$;

$$\text{i.e.}\quad v_0 = \underbrace{\begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix}}_{P} \begin{bmatrix} 500 \\ 200 \\ 100 \end{bmatrix}.$$

$$\text{So,}\quad v_2 = A^2 v_0 = (PD^2P^{-1})\left(P\begin{bmatrix} 500 \\ 200 \\ 100 \end{bmatrix}\right) = PD^2\begin{bmatrix} 500 \\ 200 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} 60 & -60 & * \\ 20 & -30 & * \\ 30 & 90 & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 500 \\ 200 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} 60 & -60 & * \\ 20 & -30 & * \\ 30 & 90 & * \end{bmatrix} \begin{bmatrix} 500 \\ 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 33{,}000 \\ 8500 \\ 19500 \end{bmatrix} \begin{matrix} \leftarrow \text{not exposed} \\ \leftarrow \text{sick} \\ \leftarrow \text{immune} \end{matrix}$$

So, 8500 ppl will be sick w/ the disease 2 yrs later.

3. <u>Recall:</u> $ax + by + cz + d = 0$ represents the equation of a plane in $\mathbb{R}^3$ with normal $\vec{n} = (a, b, c)$.

$\vec{AB} = (1, -2, 2)$, $\vec{AC} = (-1, 1, -6)$.

So, $\vec{AB}$ & $\vec{AC}$ are 2 direction vectors that define our plane.

To find the vector normal to $\vec{AB}$ & $\vec{AC}$ we should find $\vec{AB} \times \vec{AC}$.

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ -1 & 1 & -6 \end{vmatrix} = i \begin{vmatrix} -2 & 2 \\ 1 & -6 \end{vmatrix} - j \begin{vmatrix} 1 & 2 \\ -1 & -6 \end{vmatrix}$$

$$+ k \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = \begin{pmatrix} 10 \\ 4 \\ -1 \end{pmatrix}.$$

So, $10x + 4y - z + d$ should be the equation of our plane where d must be a point on the plane.

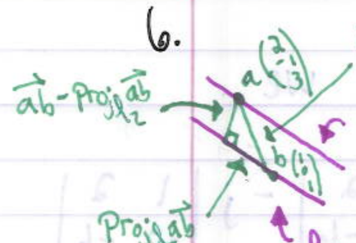
$$10(2) + 4(1) - 3 = 20 + 4 - 3 = 21.$$ ↙ plugging in $A(2,1,3)$

So
$10x + 4y - z - 21 = 0$ is the eqn of the plane passing through A, B, & C.

5. Recall: $\begin{bmatrix} \text{Volume of} \\ \text{parallelpiped} \\ \text{determined by } u, v, \& w \end{bmatrix} = |u \cdot (v \times w)|$. (see pg. 167).

$$v \times w = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}. \quad |u \cdot v \times w| = \left| \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right| = |-4+3-1| = |-2| = \boxed{2}.$$

6.



$l_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} t$

$l_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} t$

So, from the picture we can see that we're looking for $\vec{ab} - \text{proj}_{l_2}\vec{ab}$.

$\vec{ab} = (-1, 1, -2)$.

$$\text{proj}_{l_2}\vec{ab} = \frac{\vec{ab} \cdot l_2}{\|l_2\|^2} l_2 = \frac{\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}}{1+1+16} (1,-1,4)$$

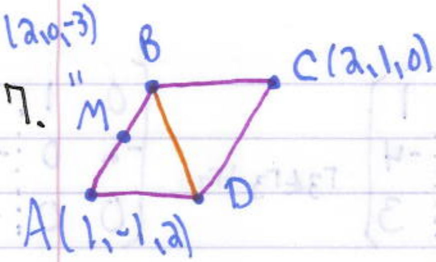
$$= \frac{-1-1-8}{18} (1,-1,4) = \left(\frac{-5}{9}, \frac{5}{9}, \frac{-20}{9} \right).$$

$$\vec{ab} - \text{proj}_{l_2}\vec{ab} = (-1, 1, -2) - \left(\frac{-5}{9}, \frac{5}{9}, \frac{-20}{9} \right)$$

$$= \left(\frac{-4}{9}, \frac{4}{9}, \frac{2}{9} \right).$$

$$\|\vec{ab} - \text{proj}_{l_2}\vec{ab}\| = \sqrt{\left(\frac{-4}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(\frac{2}{9}\right)^2} = \sqrt{\frac{16}{81} + \frac{16}{81} + \frac{4}{81}}$$

$$= \sqrt{\frac{36}{81}} = \frac{6}{9} = \boxed{\frac{2}{3}}$$

7. 

$M = A + \frac{1}{2}\vec{AB}$

$(2,0,-3) = (1,-1,2) + \frac{1}{2}(b_1 - 1, b_2 + 1, b_3 - 2)$

$(1,1,-5) = (\frac{1}{2}b_1 - \frac{1}{2}, \frac{1}{2}b_2 + \frac{1}{2}, \frac{1}{2}b_3 - 1)$

$\Rightarrow \frac{1}{2}b_1 - \frac{1}{2} = 1 \Rightarrow \frac{1}{2}b_1 = \frac{3}{2} \Rightarrow b_1 = 3.$

$\Rightarrow \frac{1}{2}b_2 + \frac{1}{2} = 1 \Rightarrow \frac{1}{2}b_2 = \frac{1}{2} \Rightarrow b_2 = 1.$

$\Rightarrow \frac{1}{2}b_3 - 1 = -5 \Rightarrow \frac{1}{2}b_3 = -4 \Rightarrow b_3 = -8.$

So, $B = (3,1,-8)$.

We can see $\vec{CD} = \vec{BA} = (-2,-2,10)$.

So, $(d_1 - 2, d_2 - 1, d_3) = (-2,-2,10)$

$\Rightarrow d_1 = 0,\ d_2 = -1,\ d_3 = 10.$

So, $D = (0,-1,10)$.

So, $\vec{BD} = (-3,-2,18)$.

10. Let $\bar{0} = \begin{pmatrix} a \\ b \end{pmatrix}$. We need $v + \bar{0} = v \ \forall v \in V$.

Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then $v + \bar{0} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} v_1 a \\ v_2 + b \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\Rightarrow v_1 a = v_1 \Rightarrow a = 1.$

$\Rightarrow v_2 + b = v_2 \Rightarrow b = 0.$

So, $\bar{0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

11. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $(-u) = \begin{pmatrix} a \\ b \end{pmatrix}$. We need $u + (-u) = \bar{0}$

$\Rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1 a \\ u_2 + b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow u_1 a = 1 \Rightarrow a = \frac{1}{u_1}.$

$\Rightarrow u_2 + b = 0 \Rightarrow b = -u_2.$

So, $(-u) = (\frac{1}{u_1}, -u_2)$.

15. $\left[\begin{array}{cc:c} 0 & 1 & 1 \\ -2 & 3 & -1 \\ 2 & -1 & 2 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 - 3r_1 \\ r_3 \leftarrow r_3 + r_1 \end{array} \left[\begin{array}{cc:c} 0 & 1 & 1 \\ -2 & 0 & -4 \\ 2 & 0 & 3 \end{array}\right] \; r_3 \leftarrow r_3 + r_2 \left[\begin{array}{cc:c} 0 & 1 & 1 \\ -2 & 0 & -4 \\ 0 & 0 & -1 \end{array}\right]$ ← No solution

$\Rightarrow \alpha_1 u + \alpha_2 v = (1,-1,2)$ has no solution
$\Rightarrow (1,-1,2) \notin \text{span}\{u,v\}$.

We can check the next two simultaneously:

$$\left[\begin{array}{cc:cc} 0 & 1 & 1 & 5 \\ -2 & 3 & 1 & 3 \\ 2 & -1 & 1 & 7 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 - 3r_1 \\ r_3 \leftarrow r_3 + r_1 \end{array} \left[\begin{array}{cc:cc} 0 & 1 & 1 & 5 \\ -2 & 0 & -2 & -12 \\ 2 & 0 & 2 & 12 \end{array}\right] r_3 \leftarrow r_3 + r_2$$

$$\left[\begin{array}{cc:cc} 0 & 1 & 1 & 5 \\ -2 & 0 & -2 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

For the first we see $\alpha_2 = 1$, $-2\alpha_1 = -2$ $\Rightarrow \alpha_1 = 1$. So, $u + v = (1,1,1)$. ✓

For the second we see $\alpha_2 = 5$, $-2\alpha_1 = -12$ $\Rightarrow \alpha_1 = 6$. So, $6u + 5v = (5,-12,0)$. ✓

So, $(1,1,1)$ & $(5,3,7) \in \text{span}\{u,v\}$.

16. $\{v,w\}$ independent $\Rightarrow \beta_1 v + \beta_2 w = 0 \Leftrightarrow (\beta_1,\beta_2) = (0,0)$.

We want $\{K_1 v + w, v + K_2 w\}$ to be independent
$\Rightarrow \alpha_1(K_1 v + w) + \alpha_2(v + K_2 w) = 0$ only has the solution $\alpha_1 = \alpha_2 = 0$.

$\alpha_1(K_1 v + w) + \alpha_2(v + K_2 w) = \alpha_1 K_1 v + \alpha_1 w + \alpha_2 v + \alpha_2 K_2 w$
$= (\alpha_1 K_1 + \alpha_2)v + (\alpha_1 + \alpha_2 K_2)w \Rightarrow \alpha_1 K_1 + \alpha_2 = 0$ (1) & $\alpha_1 + \alpha_2 K_2 = 0$ (2) (since v & w linearly independent)

$\Rightarrow \alpha_1 = -K_2 \alpha_2 \Rightarrow (-K_2\alpha_2)K_1 + \alpha_2 = 0 \Rightarrow (-K_1 K_2 + 1)\alpha_2 = 0$.

(rearrange (2); sub. $\alpha_1 = -K_2\alpha_2$ into (1))

Now, our set is <u>not</u> linearly independent if $(\alpha_1,\alpha_2) \neq (0,0)$. If

$\alpha_2 \neq 0 \Rightarrow -K_1K_2+1=0 \Rightarrow K_1K_2=1.$

So, if $K_1K_2 \neq 1 \Rightarrow \alpha_2=0$ since $(-K_1K_2+1)\alpha_2=0$

$\Rightarrow \alpha_1 = 0$ since $\alpha_1 = -K_2\underbrace{\alpha_2}_{=0}$.

So, $K_1K_2 \neq 1 \Rightarrow \alpha_1(K_1v+w)+\alpha_2(v+K_2w)=0$
only has the solution $\alpha_1=\alpha_2=0$.

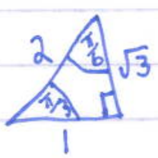
17. $\begin{bmatrix} -1 & -i & 1 \\ -i & 1 & i \\ 1 & i & -1 \end{bmatrix} \begin{matrix} r_2 \leftarrow r_2 - ir_1 \\ r_3 \leftarrow r_3 + r_1 \end{matrix} \begin{bmatrix} -1 & -i & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} r_1 \leftarrow r_1 * (-1)$

$\begin{bmatrix} 1 & i & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

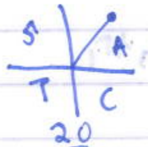
18. $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{60}$.

Let's put $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ in polar form: (i.e. want $re^{i\theta}$).

$|z| = r = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1.$



$\frac{1}{2}+\frac{\sqrt{3}}{2}i = 1\cdot\cos\theta + i\sin\theta$. So $\theta = \frac{\pi}{3}$.



So, $\frac{1}{2}+\frac{\sqrt{3}}{2}i = e^{\frac{\pi}{3}i}$. $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{60} = \left(e^{\frac{\pi}{3}i}\right)^{60}$

$= e^{\frac{60\pi}{3}i} = e^{20i\pi} = e^{0i} = \boxed{1}$. since $\theta = 20\pi = 0 + 2k\pi$ for $k=10$.

$60 \div 3 = 20$

Sample Test #2

1.

Let $A = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \end{array}$. P_{ij} is the probability the system moves from state j to state i.

So, $A = \begin{bmatrix} 0 & 2/3 & 2/3 \\ 0 & 0 & 1/3 \\ 1 & 1/3 & 0 \end{bmatrix}$.

$P_{11} = P_{22} = P_{33} = 0$ since the fox never hunts in the same place in 2 successive days.

P_{31} is the prob. fox moves from $A \to C$, so $P_{31} = 1$.

$P_{12} = 2P_{32}$ b/c P_{12} is $B \to A$ & $B \to A$ twice as likely as $B \to C$.
$P_{13} = 2P_{23}$ b/c P_{13} is $C \to A$ & $C \to A$ " " " " $C \to B$.

We need: $P_{12} + P_{32} = 1 \Rightarrow 2P_{32} + P_{32} = 1 \Rightarrow P_{32} = \frac{1}{3} \Rightarrow P_{12} = \frac{2}{3}$.
$P_{13} + 2P_{23} = 1 \Rightarrow 3P_{23} = 1 \Rightarrow P_{23} = \frac{1}{3} \Rightarrow P_{13} = \frac{2}{3}$.

We're looking for the eigenvector corresponding to $\lambda = 1$:
(i.e. the steady-state vector)

$\frac{2}{3} + \frac{2}{9} = \frac{8}{9}$

$$\left[\begin{array}{ccc|c} -1 & 2/3 & 2/3 & 0 \\ 0 & -1 & 1/3 & 0 \\ 1 & 1/3 & -1 & 0 \end{array}\right] \begin{array}{l} r_1 \leftarrow r_1 + \frac{2}{3}r_2 \\ \\ r_3 \leftarrow r_3 + \frac{1}{3}r_2 \end{array} \left[\begin{array}{ccc|c} -1 & 0 & 8/9 & 0 \\ 0 & -1 & 1/3 & 0 \\ 1 & 0 & -8/9 & 0 \end{array}\right] r_3 \leftarrow r_3 + r_1$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 8/9 & 0 \\ 0 & -1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x = \frac{8}{9}z \\ y = \frac{1}{3}z \\ z = t \end{array} \quad \begin{pmatrix} 8/9 \\ 1/3 \\ 1 \end{pmatrix} t \text{ eigenvector.}$$

Need $\frac{8}{9}t + \frac{1}{3}t + t = 1 \Rightarrow \frac{20}{9}t = 1 \Rightarrow t = \frac{9}{20}$.

$\begin{pmatrix} 8/9 \\ 1/3 \\ 1 \end{pmatrix} \frac{9}{20} = \begin{pmatrix} 8/20 \\ 3/20 \\ 9/20 \end{pmatrix} \Rightarrow$ time spent in C in long run is $\frac{9}{20}$.

2. We Know $v_n = A^n v_0$. ← system at time 0

↑ system at time n ↖ transition matrix

Here, $v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We want v_2.

$$v_2 = A^2 v_0 = \begin{bmatrix} 0 & 2/3 & 2/3 \\ 0 & 0 & 1/3 \\ 1 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 2/3 \\ 0 & 0 & 1/3 \\ 1 & 1/3 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 2/3 & 2/3 \\ 0 & 0 & 1/3 \\ 1 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix}.$$

So, probability of B on Wednesday is 1/3.

3. $X = au + bv + cw$

$$\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 2 & 2 & -1 & 6 \end{array}\right] r_3 \leftarrow r_3 + r_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 3 & 2 & 0 & 8 \end{array}\right] r_3 \leftarrow r_3 - 2r_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 10 \end{array}\right]$$

$a = 10$

$a + b = -1 \Rightarrow b = -11.$

$a + c = 2 \Rightarrow c = -8.$

9. The normal direction to the plane is $\vec{n} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, so

$l = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} t$ is the line going through P with direction $\vec{n}$. The point Q where this line intersects the plane will give us the point we want.

Our plane has eq'n $3x - y + 4z = 1$.

$y = 3x + 4z - 1.$

$x = s$

$z = r$

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} r + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$ ← vector eq'n of plane

To solve where l & the plane intersect, set them equal to each other:

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} t = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} r + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} r + \begin{pmatrix} -3 \\ 1 \\ -4 \end{pmatrix} t = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} s \\ r \\ t \end{bmatrix}$$

s r t

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 3 & 4 & 1 & 2 \\ 0 & 1 & -4 & -3 \end{array}\right] r_2 \leftarrow r_2 - 3r_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 4 & 10 & -4 \\ 0 & 1 & -4 & -3 \end{array}\right] r_2 \leftarrow r_2 - 4r_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 0 & 26 & 8 \\ 0 & 1 & -4 & -3 \end{array}\right]$$

$26t = 8 \Rightarrow t = \frac{8}{26} = \frac{4}{13}$

$s = 3t + 2 = \frac{12}{13} + 2 = \frac{38}{13}$

$r = 4t - 3 = \frac{16}{13} - 3 = \frac{16}{13} - \frac{39}{13} = \frac{-23}{13}$

$38 \times 3 = 114$ $\qquad$ $23 \times 4 = 92$

$\frac{114}{13} - \frac{92}{13} - \frac{13}{13} = \frac{9}{13}$

$$\begin{pmatrix}1\\3\\0\end{pmatrix} s + \begin{pmatrix}0\\4\\1\end{pmatrix} r + \begin{pmatrix}0\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix} \frac{38}{13} + \begin{pmatrix}0\\4\\1\end{pmatrix} \frac{-23}{13} + \begin{pmatrix}0\\-1\\0\end{pmatrix}$$

$$= \left(\frac{38}{13}, \frac{9}{13}, \frac{-23}{13}\right).$$