Math 2Z03 - Tutorial # 7

Nov. 2nd, 3rd, 4th, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

Tutorial #7:

- Quick Linear Algebra Review

Tutorial #7:

- Quick Linear Algebra Review
- 3.5 Variation of Parameters

Tutorial #7:

- Quick Linear Algebra Review
- 3.5 Variation of Parameters
- 3.8 Linear Models: IVP's
  - Spring/Mass Systems
  - LRC-Series Circuits

Quick Linear Algebra Review

- **Equivalent Statements:**

Quick Linear Algebra Review

- **Equivalent Statements:**
- We know several equivalent statements, where A is a $n \times n$ matrix:

Quick Linear Algebra Review

- **Equivalent Statements:**
- We know several equivalent statements, where A is a $n \times n$ matrix:

(a) A is invertible.

Quick Linear Algebra Review

- **Equivalent Statements:**
- We know several equivalent statements, where A is a $n \times n$ matrix:

(a) A is invertible.
(b) $Ax = 0$ has only the trivial solution.

Quick Linear Algebra Review

- **Equivalent Statements:**
- We know several equivalent statements, where A is a $n \times n$ matrix:

(a) A is invertible.
(b) $Ax = 0$ has only the trivial solution.
(c) The reduced row echelon form of A is I_n.

Quick Linear Algebra Review

- **Equivalent Statements:**
- We know several equivalent statements, where A is a $n \times n$ matrix:
  (a) A is invertible.
  (b) $Ax = 0$ has only the trivial solution.
  (c) The reduced row echelon form of A is I_n.
  (d) A is expressible as the product of elementary matrices.

Quick Linear Algebra Review

- **Equivalent Statements:**
- We know several equivalent statements, where A is a $n \times n$ matrix:
  (a) A is invertible.
  (b) $Ax = 0$ has only the trivial solution.
  (c) The reduced row echelon form of A is I_n.
  (d) A is expressible as the product of elementary matrices.
  (e) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

Quick Linear Algebra Review

- **Properties of Determinants:**

Quick Linear Algebra Review

- **Properties of Determinants:**

(a) $\det(A) = \det(A^T)$

Quick Linear Algebra Review

- **Properties of Determinants:**
  (a) $\det(A) = \det(A^T)$
  (b) $\det(AB) = \det(A)\det(B)$

Quick Linear Algebra Review

- **Properties of Determinants:**
  (a) $\det(A) = \det(A^T)$
  (b) $\det(AB) = \det(A)\det(B)$
  (c) $\det(kA) = k^n \det(A)$, where $k \in \mathbb{R}$, and A is a $n \times n$ matrix.

Quick Linear Algebra Review

- **Properties of Determinants:**
  (a) $\det(A) = \det(A^T)$
  (b) $\det(AB) = \det(A)\det(B)$
  (c) $\det(kA) = k^n \det(A)$, where $k \in \mathbb{R}$, and A is a $n \times n$ matrix.
  (d) $\det(A) \neq 0 \Leftrightarrow A$ is invertible.

Quick Linear Algebra Review

- **Properties of Determinants:**
  (a) $\det(A) = \det(A^T)$
  (b) $\det(AB) = \det(A)\det(B)$
  (c) $\det(kA) = k^n \det(A)$, where $k \in \mathbb{R}$, and A is a $n \times n$ matrix.
  (d) $\det(A) \neq 0 \Leftrightarrow A$ is invertible.
  (e) $\det(A^{-1}) = \frac{1}{\det(A)}$.

Quick Linear Algebra Review

- **Properties of Determinants:**
  (a) $\det(A) = \det(A^T)$
  (b) $\det(AB) = \det(A)\det(B)$
  (c) $\det(kA) = k^n \det(A)$, where $k \in \mathbb{R}$, and A is a $n \times n$ matrix.
  (d) $\det(A) \neq 0 \Leftrightarrow A$ is invertible.
  (e) $\det(A^{-1}) = \frac{1}{\det(A)}$.

- **Inverse Properties:**

Quick Linear Algebra Review

- **Properties of Determinants:**
  (a) $\det(A) = \det(A^T)$
  (b) $\det(AB) = \det(A)\det(B)$
  (c) $\det(kA) = k^n \det(A)$, where $k \in \mathbb{R}$, and A is a $n \times n$ matrix.
  (d) $\det(A) \neq 0 \Leftrightarrow A$ is invertible.
  (e) $\det(A^{-1}) = \frac{1}{\det(A)}$.

- **Inverse Properties:**
  (a) $(A^{-1})^{-1} = A$

Quick Linear Algebra Review

- **Properties of Determinants:**
  (a) $\det(A) = \det(A^T)$
  (b) $\det(AB) = \det(A)\det(B)$
  (c) $\det(kA) = k^n \det(A)$, where $k \in \mathbb{R}$, and A is a $n \times n$ matrix.
  (d) $\det(A) \neq 0 \Leftrightarrow A$ is invertible.
  (e) $\det(A^{-1}) = \frac{1}{\det(A)}$.

- **Inverse Properties:**
  (a) $(A^{-1})^{-1} = A$
  (b) $(AB)^{-1} = B^{-1}A^{-1}$

Quick Linear Algebra Review

- **Properties of Determinants:**
  (a) $\det(A) = \det(A^T)$
  (b) $\det(AB) = \det(A)\det(B)$
  (c) $\det(kA) = k^n \det(A)$, where $k \in \mathbb{R}$, and A is a $n \times n$ matrix.
  (d) $\det(A) \neq 0 \Leftrightarrow A$ is invertible.
  (e) $\det(A^{-1}) = \frac{1}{\det(A)}$.

- **Inverse Properties:**
  (a) $(A^{-1})^{-1} = A$
  (b) $(AB)^{-1} = B^{-1}A^{-1}$
  (c) $(A^T)^{-1} = (A^{-1})^T$

Quick Linear Algebra Review

- **Transpose Properties:**

Quick Linear Algebra Review

- **Transpose Properties:**

(a) $(A^T)^T = A$

Quick Linear Algebra Review

- **Transpose Properties:**
  (a) $(A^T)^T = A$
  (b) $(A+B)^T = A^T + B^T$

Quick Linear Algebra Review

- **Transpose Properties:**
  (a) $(A^T)^T = A$
  (b) $(A+B)^T = A^T + B^T$
  (c) $(AB)^T = B^T A^T$

Quick Linear Algebra Review

- **Transpose Properties:**
  (a) $(A^T)^T = A$
  (b) $(A+B)^T = A^T + B^T$
  (c) $(AB)^T = B^T A^T$
  (d) $(kA)^T = kA^T$ for a scalar k

Quick Linear Algebra Review

- **Transpose Properties:**
  (a) $(A^T)^T = A$
  (b) $(A+B)^T = A^T + B^T$
  (c) $(AB)^T = B^T A^T$
  (d) $(kA)^T = kA^T$ for a scalar k

- The **rank** of an $m \times n$ matrix A, $\text{rank}(A)$, is the maximum number of linearly independent row vectors in A.

Quick Linear Algebra Review

- **Transpose Properties:**
  (a) $(A^T)^T = A$
  (b) $(A+B)^T = A^T + B^T$
  (c) $(AB)^T = B^T A^T$
  (d) $(kA)^T = kA^T$ for a scalar k

- The **rank** of an $m \times n$ matrix A, $\text{rank}(A)$, is the maximum number of linearly independent row vectors in A.

- In other words, the reduced row echelon form of A will have exactly $\text{rank}(A)$ nonzero rows.

Quick Linear Algebra Review

- **1.** Solve the linear system of equations

$$\begin{aligned} 3x + 2y + z &= 1 \\ 5x + 4y + 2z &= -1. \end{aligned}$$

Quick Linear Algebra Review

- **1.** Solve the linear system of equations

$$\begin{aligned} 3x + 2y + z &= 1 \\ 5x + 4y + 2z &= -1. \end{aligned}$$

- **Recall:** For any system of equations, IF a solution exists, then # free variables = # of columns - # of leading 1's.

Quick Linear Algebra Review

- **1.** Solve the linear system of equations

$$\begin{aligned} 3x+2y+z &= 1 \\ 5x+4y+2z &= -1. \end{aligned}$$

- **Recall:** For any system of equations, IF a solution exists, then # free variables = # of columns - # of leading 1's.

- **2.** Find the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix}$ using row operations.

Quick Linear Algebra Review

- **3.** Find the inverse of $\begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{pmatrix}$ using the adjoint method.

Quick Linear Algebra Review

- **3.** Find the inverse of $\begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{pmatrix}$ using the adjoint method.
- **4.** Find the determinant of $\begin{pmatrix} 3 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 5 & 3 \end{pmatrix}$.

3.6 Cauchy-Euler Equations

- A linear DE of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x)$$

is known as a **Cauchy-Euler equation**.

3.6 Cauchy-Euler Equations

- **Method of Solution # 1:**

3.6 Cauchy-Euler Equations

- **Method of Solution # 1:**
  1. Solve the corresponding homogeneous equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$$

by trying a solution of the form $y = x^m$. **i.e.** Plug $y = x^m$ into the equation.

3.6 Cauchy-Euler Equations

- **Method of Solution # 1:**

1. Solve the corresponding homogeneous equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$$

   by trying a solution of the form $y = x^m$. **i.e.** Plug $y = x^m$ into the equation.
2. Find all roots to the equation on $(0, \infty)$. **i.e.** Factor x^m out and find the roots of the corresponding auxiliary equation.

3.6 Cauchy-Euler Equations

- **Method of Solution # 1:**
  1. Solve the corresponding homogeneous equation

     $$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$$

     by trying a solution of the form $y = x^m$. **i.e.** Plug $y = x^m$ into the equation.
  2. Find all roots to the equation on $(0, \infty)$. **i.e.** Factor x^m out and find the roots of the corresponding auxiliary equation.
  3. Analogous to 3.3, we have three cases:

3.6 Cauchy-Euler Equations

- **Method of Solution # 1:**
  1. Solve the corresponding homogeneous equation

     $$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$$

     by trying a solution of the form $y = x^m$. **i.e.** Plug $y = x^m$ into the equation.
  2. Find all roots to the equation on $(0, \infty)$. **i.e.** Factor x^m out and find the roots of the corresponding auxiliary equation.
  3. Analogous to 3.3, we have three cases:
     a) If there are k distinct roots $m_1, \ldots, m_k$, the general solution contains a linear combination of the functions $x^{m_1}, \ldots, x^{m_k}$.

3.6 Cauchy-Euler Equations

- **Method of Solution # 1:**
  1. Solve the corresponding homogeneous equation

     $$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$$

     by trying a solution of the form $y = x^m$. **i.e.** Plug $y = x^m$ into the equation.
  2. Find all roots to the equation on $(0, \infty)$. **i.e.** Factor x^m out and find the roots of the corresponding auxiliary equation.
  3. Analogous to 3.3, we have three cases:
     a) If there are k distinct roots $m_1, \ldots, m_k$, the general solution contains a linear combination of the functions $x^{m_1}, \ldots, x^{m_k}$.
     b) If there is a root m of multiplicity r, then the general solution contains a linear combination of the functions $x^m, x^m \ln x, x^m (\ln x)^2, \ldots, x^m (\ln x)^{r-1}$.

3.6 Cauchy-Euler Equations

- **Method of Solution # 1:**

1. Solve the corresponding homogeneous equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$$

   by trying a solution of the form $y = x^m$. **i.e.** Plug $y = x^m$ into the equation.
2. Find all roots to the equation on $(0, \infty)$. **i.e.** Factor x^m out and find the roots of the corresponding auxiliary equation.
3. Analogous to 3.3, we have three cases:
   a) If there are k distinct roots $m_1, \ldots, m_k$, the general solution contains a linear combination of the functions $x^{m_1}, \ldots, x^{m_k}$.
   b) If there is a root m of multiplicity r, then the general solution contains a linear combination of the functions $x^m, x^m \ln x, x^m (\ln x)^2, \ldots, x^m (\ln x)^{r-1}$.
   c) If there is a complex conjugate pair $\alpha \pm \beta i$, then the general solution contains the linear combination $x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$, for constants c_1, c_2.

3.6 Cauchy-Euler Equations

- **Method of Solution # 1:**

1. Solve the corresponding homogeneous equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$$

   by trying a solution of the form $y = x^m$. **i.e.** Plug $y = x^m$ into the equation.
2. Find all roots to the equation on $(0, \infty)$. **i.e.** Factor x^m out and find the roots of the corresponding auxiliary equation.
3. Analogous to 3.3, we have three cases:
   a) If there are k distinct roots $m_1, \ldots, m_k$, the general solution contains a linear combination of the functions $x^{m_1}, \ldots, x^{m_k}$.
   b) If there is a root m of multiplicity r, then the general solution contains a linear combination of the functions $x^m, x^m \ln x, x^m (\ln x)^2, \ldots, x^m (\ln x)^{r-1}$.
   c) If there is a complex conjugate pair $\alpha \pm \beta i$, then the general solution contains the linear combination $x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$, for constants c_1, c_2.
4. Use Variation of Parameters to find a particular solution to the original nonhomogeneous equation.

3.6 Cauchy-Euler Equations

- **Method of Solution # 2: (for 2nd-order Cauchy Euler Equations on $(0, \infty)$)**

3.6 Cauchy-Euler Equations

- **Method of Solution # 2: (for 2nd-order Cauchy Euler Equations on $(0, \infty)$)**
  1. Make the substitution $x = e^t$. Using this substitution and the chain rule, the corresponding homogeneous Cauchy-Euler equation

$$a_2 x^2 y'' + a_1 x y' + a_0 y = 0$$

  is transformed into the linear DE with constant coefficients

$$a_2 \frac{d^2 y(t)}{dt^2} + (a_1 - a_2) \frac{dy(t)}{dt} + a_0 y(t) = 0.$$

3.6 Cauchy-Euler Equations

- **Method of Solution # 2: (for 2nd-order Cauchy Euler Equations on $(0, \infty)$)**
  1. Make the substitution $x = e^t$. Using this substitution and the chain rule, the corresponding homogeneous Cauchy-Euler equation

     $$a_2 x^2 y'' + a_1 x y' + a_0 y = 0$$

     is transformed into the linear DE with constant coefficients

     $$a_2 \frac{d^2 y(t)}{dt^2} + (a_1 - a_2) \frac{dy(t)}{dt} + a_0 y(t) = 0.$$

  2. Solve this new DE using the methods from §3.3.

3.6 Cauchy-Euler Equations

- **Method of Solution # 2: (for 2nd-order Cauchy Euler Equations on $(0,\infty)$)**
  1. Make the substitution $x = e^t$. Using this substitution and the chain rule, the corresponding homogeneous Cauchy-Euler equation

     $$a_2x^2y'' + a_1xy' + a_0y = 0$$

     is transformed into the linear DE with constant coefficients

     $$a_2\frac{d^2y(t)}{dt^2} + (a_1 - a_2)\frac{dy(t)}{dt} + a_0y(t) = 0.$$

  2. Solve this new DE using the methods from §3.3.
  3. Switch back to the original variables $t = \ln x$.

3.6 Cauchy-Euler Equations

- **Method of Solution # 2: (for 2nd-order Cauchy Euler Equations on $(0,\infty)$)**
  1. Make the substitution $x = e^t$. Using this substitution and the chain rule, the corresponding homogeneous Cauchy-Euler equation
  $$a_2 x^2 y'' + a_1 x y' + a_0 y = 0$$
  is transformed into the linear DE with constant coefficients
  $$a_2 \frac{d^2 y(t)}{dt^2} + (a_1 - a_2)\frac{dy(t)}{dt} + a_0 y(t) = 0.$$
  2. Solve this new DE using the methods from §3.3.
  3. Switch back to the original variables $t = \ln x$.
  4. For a nonhomogenous DE, you can use this substitution on the RHS too, and solve using undetermined coefficients, then sub. back. OR, First go back to the original variables, then find a particular solution using Variation of Parameters.

3.6 Cauchy-Euler Equations

- **5.** Find the general solution to the following:

3.6 Cauchy-Euler Equations

- **5.** Find the general solution to the following:

  a) $x^2y'' + 10xy' + 8y = x^2$

3.6 Cauchy-Euler Equations

- **5.** Find the general solution to the following:

  a) $x^2y'' + 10xy' + 8y = x^2$

  b) $x^2y'' + xy' + 4y = 0$

3.6 Cauchy-Euler Equations

- **5.** Find the general solution to the following:

  a) $x^2y'' + 10xy' + 8y = x^2$

  b) $x^2y'' + xy' + 4y = 0$

  c) $x^3y''' - 6y = 0$

3.6 Cauchy-Euler Equations

- **5.** Find the general solution to the following:
  a) $x^2y'' + 10xy' + 8y = x^2$
  b) $x^2y'' + xy' + 4y = 0$
  c) $x^3y''' - 6y = 0$
  d) $x^2y'' - xy' + y = 0$

3.8 Spring/Mass Systems

- **6.** When a mass of 2kg is attached to a spring whose constant is 32N/m, it comes to rest in the equilibrium position. Starting at $t = 0$, a force equal to $f(t) = 68e^{-2t}\cos(4t)$ is applied to the system.

  a) Find the equation of motion in the absence of damping.

  b) What is the amplitude of vibrations after a very long time?