

Math 2Z03 - Tutorial # 7



Nov. 2nd, 3rd, 4th, 2015

Tutorial Info:

- **Tutorial Website:** <http://ms.mcmaster.ca/~dedieula/2Z03.html>
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)



Tutorial #7:

- Quick Linear Algebra Review



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- 3.5 Variation of Parameters



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- Quick Linear Algebra Review
- 3.5 Variation of Parameters
- 3.8 Linear Models: IVP's
 - Spring/Mass Systems
 - LRC-Series Circuits



Quick Linear Algebra Review

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 - (d) A is expressible as the product of elementary matrices.
 - (e) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b .



Quick Linear Algebra Review

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- (c) $(A^T)^{-1} = (A^{-1})^T$



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- (d) $(kA)^T = kA^T$ for a scalar k



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- The **rank** of an $m \times n$ matrix A , $\text{rank}(A)$, is the maximum number of linearly independent row vectors in A .
- In other words, the reduced row echelon form of A will have exactly $\text{rank}(A)$ nonzero rows.



Quick Linear Algebra Review

- 1. Solve the linear system of equations

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- **Recall:** For any system of equations, IF a solution exists, then # free variables = # of columns - # of leading 1's.

- 2. Find the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix}$ using row operations.



Quick Linear Algebra Review

- 3. Find the inverse of $\begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{pmatrix}$ using the adjoint method.



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- **3.** Find the inverse of $\begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{pmatrix}$ using the adjoint method.
- **4.** Find the determinant of $\begin{pmatrix} 3 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 5 & 3 \end{pmatrix}$.



3.6 Cauchy-Euler Equations

- A linear DE of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x)$$

is known as a **Cauchy-Euler equation**.



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- **Method of Solution # 1:**



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1. Solve the corresponding homogeneous equation

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2. Find all roots to the equation on $(0, \infty)$. **i.e.** Factor x^m out and find the roots of the corresponding auxiliary equation.



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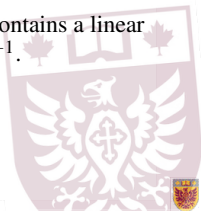
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 - a) If there are k distinct roots m_1, \dots, m_k , the general solution contains a linear combination of the functions x^{m_1}, \dots, x^{m_k} .
 - b) If there is a root m of multiplicity r , then the general solution contains a linear combination of the functions $x^m, x^m \ln x, x^m (\ln x)^2, \dots, x^m (\ln x)^{r-1}$.



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 - c) If there is a complex conjugate pair $\alpha \pm \beta i$, then the general solution contains the linear combination $x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$, for constants c_1, c_2 .



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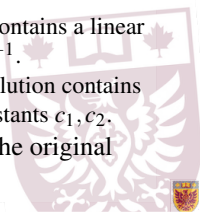
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4. Use Variation of Parameters to find a particular solution to the original nonhomogeneous equation.



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- **Method of Solution # 2: (for 2nd-order Cauchy Euler Equations on $(0, \infty)$)**



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■ **Method of Solution # 2: (for 2nd-order Cauchy Euler Equations on $(0, \infty)$)**

1. Make the substitution $x = e^t$. Using this substitution and the chain rule, the corresponding homogeneous Cauchy-Euler equation

$$a_2x^2y'' + a_1xy' + a_0y = 0$$

is transformed into the linear DE with constant coefficients

$$a_2 \frac{d^2y(t)}{dt^2} + (a_1 - a_2) \frac{dy(t)}{dt} + a_0y(t) = 0.$$



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2. Solve this new DE using the methods from §3.3.
3. Switch back to the original variables $t = \ln x$.
4. For a nonhomogenous DE, you can use this substitution on the RHS too, and solve using undetermined coefficients, then sub. back. OR, First go back to the original variables, then find a particular solution using Variation of Parameters.



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d) $x^2y'' - xy' + y = 0$



3.8 Spring/Mass Systems

- **6.** When a mass of 2kg is attached to a spring whose constant is 32N/m, it comes to rest in the equilibrium position. Starting at $t = 0$, a force equal to $f(t) = 68e^{-2t}\cos(4t)$ is applied to the system.
 - a) Find the equation of motion in the absence of damping.
 - b) What is the amplitude of vibrations after a very long time?

