Math 2Z03 - Tutorial # 6

Oct. 26th, 27th, 28th, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

Tutorial #6:

- 3.4 Undetermined Coefficients
- 3.5 Variation of Parameters

3.4 Undetermined Coefficients

- **Undetermined Coefficients:** a method of solution for finding a particular solution to a linear differential equation with *constant coefficients*

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x),$$

where $g(x)$ is a polynomial, exponential e^{ax}, sine, cosine, or some sum/product of these function.

3.4 Undetermined Coefficients

- **Method of Solution:**
  1. Find the general solution, y_c, to the associated homogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x).$$

  2. Choose a trial particular solution, y_{p_i} for each term in $g(x)$.
  3. If any y_{p_i} contains terms that duplicate terms in y_c, then multiply y_{p_i} by x^n, where n is the smallest possible integer that eliminate the duplication.
  4. Plug the sum of the terms found in Step 3, y_p, into the original DE and solve for the undetermined coefficients.
  5. The general solution is $y = y_c + y_p$.

3.4 Undetermined Coefficients

- **Trial Particular Solutions:**
  - A polynomial of degree k corresponds to a trial solution $a_0 + a_1 x + \cdots + a_k x^k$.
  - e^{mx} corresponds to trial solution Ae^{mx}.
  - $\sin(mx)$ and $\cos(mx)$ both correspond to the trial solution $A\cos(mx) + B\sin(mx)$.
  - If you have a product of the above, then take the corresponding product of trial solutions.
    e.g. $xe^{3x}\cos(4x)$ corresponds to trial solution
    $(a_0 + a_1 x)(Ae^{3x})(B\cos(4x) + C\sin(4x)) =$
    $(c_0 + c_1 x)(e^{3x})(B\cos(4x) + C\sin(4x))$.

3.4 Undetermined Coefficients

- **1.** For each of the following, the general solution of the associated homogenous equation is given. What form will its particular solution have?
  a) $y'' + 3y = -48x^2e^{3x}$; $y_c = c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)$
  b) $y''' - 6y'' = 3 - \cos x$; $y_c = c_1 + c_2x + c_3e^{6x}$
  c) $y'' - y' + \frac{1}{4}y = 3 + e^{\frac{x}{2}}$; $y_c = c_1e^{\frac{x}{2}} + c_2xe^{\frac{x}{2}}$

- **2.** Find the general solution of the differential equations in #1.

3.5 Variation of Parameters

- **Variation of Parameters:** a method of solution for finding a particular solution to a linear differential equation

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x).$$

To use it, we must already have a general solution for the corresponding homogeneous equation

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

- **Advantages Over Method of Undetermined Coefficients:**
  - For Undetermined Coefficients, we require that $g(x)$ is a sum/product of polynomials, e^{ax}'s, sines, and cosines. For Variation of Parameters, we require no such restrictions on $g(x)$.
  - Undetermined Coefficients requires that the linear DE has constant coefficients, whereas Variation of Parameters does not.

3.5 Variation of Parameters

- **Method of Solution:** I write the method of solution for second-order linear DE's, but this method naturally generalizes for higher order DE's.

1. Find the general solution, $y_c = c_1 y_1 + c_2 y_2$, for corresponding homogeneous equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$.
2. Compute Wronskian $W(y_1(x), y_2(x))$.
3. Put equation in standard form $y'' + P(x)y' + Q(x)y = f(x)$.
4. Compute $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$.
5. Find $u_1 := \int \frac{W_1}{W} dx$ and $u_2 := \int \frac{W_2}{W} dx$.
6. A particular solution is $y_p = u_1 y_1 + u_2 y_2$, and the general solution is $y = y_c + y_p$.

3.5 Variation of Parameters

- **3.** Find the general solution of

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{\frac{3}{2}},$$

where $y_1 = x^{-\frac{1}{2}}\cos x$ and $y_2 = x^{-\frac{1}{2}}\sin x$ and linearly independent solutions of the associated homogeneous DE on $(0, \infty)$.

- **4.** Solve $y''' + y' = \tan x$.