Math 2Z03 - Tutorial # 6

Oct. 26th, 27th, 28th, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

Tutorial #6:

- 3.4 Undetermined Coefficients

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- 3.5 Variation of Parameters

3.4 Undetermined Coefficients

- **Undetermined Coefficients:** a method of solution for finding a particular solution to a linear differential equation with *constant coefficients*

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x),$$

where $g(x)$ is a polynomial, exponential e^{ax}, sine, cosine, or some sum/product of these function.

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  5. The general solution is $y = y_c + y_p$.

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  - If you have a product of the above, then take the corresponding product of trial solutions.
    e.g. $xe^{3x}\cos(4x)$ corresponds to trial solution
    $(a_0 + a_1 x)(Ae^{3x})(B\cos(4x) + C\sin(4x)) =$
    $(c_0 + c_1 x)(e^{3x})(B\cos(4x) + C\sin(4x))$.

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- **2.** Find the general solution of the differential equations in #1.

3.5 Variation of Parameters

- **Variation of Parameters:** a method of solution for finding a particular solution to a linear differential equation

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To use it, we must already have a general solution for the corresponding homogeneous equation

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  - For Undetermined Coefficients, we require that $g(x)$ is a sum/product of polynomials, e^{ax}'s, sines, and cosines. For Variation of Parameters, we require no such restrictions on $g(x)$.
  - Undetermined Coefficients requires that the linear DE has constant coefficients, whereas Variation of Parameters does not.

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4. Compute $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$.

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5. Find $u_1 := \int \frac{W_1}{W} dx$ and $u_2 := \int \frac{W_2}{W} dx$.
6. A particular solution is $y_p = u_1 y_1 + u_2 y_2$, and the general solution is $y = y_c + y_p$.

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- **3.** Find the general solution of

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{\frac{3}{2}},$$

where $y_1 = x^{-\frac{1}{2}}\cos x$ and $y_2 = x^{-\frac{1}{2}}\sin x$ and linearly independent solutions of the associated homogeneous DE on $(0, \infty)$.

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- **4.** Solve $y''' + y' = \tan x$.