Math 2Z03 - Tutorial # 4

Oct. 5th, 6th, 7th, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

Tutorial #4:

- 3.1 Theory of Linear Equations
  - Linearly Independence
  - Wronskian
  - Fundamental Set of Solutions
  - General Solution
- 3.3 Homogeneous Linear Equations with Constant Coefficients
  - Finding Rational Roots
  - Finding Complex Roots

3.1 Theory of Linear Equations

- **1.** Is the set of functions $\{1+x, x, x^2\}$ linearly independent on $(-\infty, \infty)$?
- **Recall:** A set of functions $f_1(x), \ldots, f_n(x)$ are **linearly independent** on an interval I if $c_1 f_1(x) + \cdots + c_n f_n(x) = 0$ for all x in $I \iff c_1 = \cdots c_n = 0$.
- **Criterion for Linear Independent Solutions:** Let $y_1, \ldots, y_n$ be solutions of a homogeneous linear n-th order DE on an interval I. Then this set of solutions is linearly independent on I if and only if the Wronskian $W(y_1, \ldots, y_n) \neq 0$ for every x in I.

3.1 Theory of Linear Equations

- **2.** Suppose f_1, f_2, and f_3 are solutions to a second-order linear homogeneous differential equation. Is $\{f_1, f_2, f_3\}$ a fundamental set of solutions?

- **Recall:** A basis for the space of solutions of an n-th order homogenous linear equation $a_n y^{(n)} + \cdots + a_1 y' + a_0 = 0$ is called a **fundamental set of solutions**.

  We know the dimension of the solution space is n, so to find a basis, it suffices to find n linearly independent solutions.

3.1 Theory of Linear Equations

- **3.** The functions e^t and te^t satisfy the differential equation $y'' - 2y' + y = 0$. Is $y = c_1 e^t + c_2 te^t$ a general solution of this differential equation?

- **Recall:** If $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions on I for an n-th order linear DE, then the **general solution** on I is $y = c_1 y_1 + \cdots c_n y_n$, where the c_i are arbitrary constants.

3.3 Homogenous Linear Equations with Constant Coefficients:

- **4.** Find the general solution of $y^{(3)} + 8y = 0$.

- **Recall:** To solve, first we plug $y = e^{mx}$ into the equation and find the roots of the corresponding auxiliary equation.

- *We can go about finding the roots of this auxiliary equation in a variety of ways.*

3.3 Homogenous Linear Equations with Constant Coefficients:

- **Method 1:** Find one root d, then divide m^3+8 by $(m-d)$.
- By inspection, we can see that $m=-2$ is a root. A more systematic approach is given by the following:
- **Rational Roots Test (pg. 122):** If $m_1 = \frac{p}{q}$ is a rational root (expressed in lowest terms) of an auxiliary equation with integer coefficients

$$a_n m^n + \ldots + a_1 m + a_0,$$

then p is a factor of a_0 and q is a factor of a_n.

3.3 Homogenous Linear Equations with Constant Coefficients:

- **Method 2: Finding roots over $\mathbb{C}$:** Given auxiliary equation

$$a_n m^n + \ldots + a_1 m + a_0,$$

we know there will be n roots over the complex numbers (with some multiplicity). By working in polar coordinates, we can find these roots $re^{i\theta}$ (think back to Math 1ZC3/1B03).

- **Exercise** Find the five roots of $m^5 + 32 = 0$, and use this to solve $y^{(5)} + 32y = 0$.

3.3 Homogenous Linear Equations with Constant Coefficients:

- **Recall:**
  - If there are j distinct roots $m_1, \ldots, m_j$ then the general solution contains a linear combination of $e^{m_1 x}, \ldots, e^{m_j x}$.
  - If m_1 is a root of multiplicity q, then then general solution contains a lineae combination of $e^{m_1 x}$, $xe^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{q-1} e^{m_1 x}$.
  - Given complex roots $m_1 = \alpha + \beta$ and $m_2 = \alpha - \beta$, using Euler's formula, it's always possible to write $c_1 e^{m_1 x} + c_2 e^{m_2 x}$ as $e^{\alpha x}[k_1 \cos(\beta x) + k_2 \sin(\beta x)]$, for constants c_1, c_2, k_1, k_2.

3.3 Homogenous Linear Equations with Constant Coefficients:

- **5.** Solve the IVP $y'' + 2y' = 0$, $y(0) = 1$, $y'(0) = 1$.

- **6.** Find a general solution for $6y^{(4)} - y''' + 4y'' - y' - 2y = 0$.