

# Math 2Z03 - Tutorial # 4



Oct. 5th, 6th, 7th, 2015

## Tutorial Info:

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- **Tutorial Website:** <http://ms.mcmaster.ca/~dedieula/2Z03.html>
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)



# Tutorial #4:

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- 3.1 Theory of Linear Equations
  - Linearly Independence
  - Wronskian
  - Fundamental Set of Solutions
  - General Solution
  
- 3.3 Homogeneous Linear Equations with Constant Coefficients
  - Finding Rational Roots
  - Finding Complex Roots



## 3.1 Theory of Linear Equations

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- **1.** Is the set of functions  $\{1 + x, x, x^2\}$  linearly independent on  $(-\infty, \infty)$ ?
- **Recall:** A set of functions  $f_1(x), \dots, f_n(x)$  are **linearly independent** on an interval  $I$  if  $c_1f_1(x) + \dots + c_nf_n(x) = 0$  for all  $x$  in  $I \iff c_1 = \dots = c_n = 0$ .
- **Criterion for Linear Independent Solutions:** Let  $y_1, \dots, y_n$  be solutions of a homogeneous linear  $n$ -th order DE on an interval  $I$ . Then this set of solutions is linearly independent on  $I$  if and only if the Wronskian  $W(y_1, \dots, y_n) \neq 0$  for every  $x$  in  $I$ .



## 3.1 Theory of Linear Equations

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- **2.** Suppose  $f_1, f_2$ , and  $f_3$  are solutions to a second-order linear homogeneous differential equation. Is  $\{f_1, f_2, f_3\}$  a fundamental set of solutions?
- **Recall:** A basis for the space of solutions of an  $n$ -th order homogeneous linear equation  $a_n y^{(n)} + \cdots + a_1 y' + a_0 = 0$  is called a **fundamental set of solutions**.

*We know the dimension of the solution space is  $n$ , so to find a basis, it suffices to find  $n$  linearly independent solutions.*



## 3.1 Theory of Linear Equations

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- **3.** The functions  $e^t$  and  $te^t$  satisfy the differential equation  $y'' - 2y' + y = 0$ . Is  $y = c_1e^t + c_2te^t$  a general solution of this differential equation?
- **Recall:** If  $\{y_1, \dots, y_n\}$  is a fundamental set of solutions on  $I$  for an  $n$ -th order linear DE, then the **general solution** on  $I$  is  $y = c_1y_1 + \dots + c_ny_n$ , where the  $c_i$  are arbitrary constants.



## 3.3 Homogenous Linear Equations with Constant Coefficients:

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- 4. Find the general solution of  $y^{(3)} + 8y = 0$ .
- **Recall:** To solve, first we plug  $y = e^{mx}$  into the equation and find the roots of the corresponding auxiliary equation.
- *We can go about finding the roots of this auxiliary equation in a variety of ways.*



## 3.3 Homogenous Linear Equations with Constant Coefficients:

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- **Method 1:** Find one root  $d$ , then divide  $m^3 + 8$  by  $(m - d)$ .
- By inspection, we can see that  $m = -2$  is a root. A more systematic approach is given by the following:
- **Rational Roots Test (pg. 122):** If  $m_1 = \frac{p}{q}$  is a rational root (expressed in lowest terms) of an auxiliary equation with integer coefficients

$$a_n m^n + \dots + a_1 m + a_0,$$

then  $p$  is a factor of  $a_0$  and  $q$  is a factor of  $a_n$ .





## 3.3 Homogenous Linear Equations with Constant Coefficients:

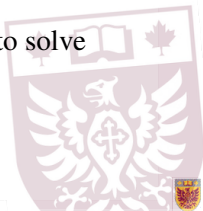
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- **Method 2: Finding roots over  $\mathbb{C}$ :** Given auxiliary equation

$$a_n m^n + \dots + a_1 m + a_0,$$

we know there will be  $n$  roots over the complex numbers (with some multiplicity). By working in polar coordinates, we can find these roots  $re^{i\theta}$  (think back to Math 1ZC3/1B03).

- **Exercise** Find the five roots of  $m^5 + 32 = 0$ , and use this to solve  $y^{(5)} + 32y = 0$ .



## 3.3 Homogenous Linear Equations with Constant Coefficients:

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### ■ Recall:

- If there are  $j$  distinct roots  $m_1, \dots, m_j$  then the general solution contains a linear combination of  $e^{m_1x}, \dots, e^{m_jx}$ .
- If  $m_1$  is a root of multiplicity  $q$ , then the general solution contains a linear combination of  $e^{m_1x}, xe^{m_1x}, x^2e^{m_1x}, \dots, x^{q-1}e^{m_1x}$ .
- Given complex roots  $m_1 = \alpha + \beta$  and  $m_2 = \alpha - \beta$ , using Euler's formula, it's always possible to write  $c_1e^{m_1x} + c_2e^{m_2x}$  as  $e^{\alpha x} [k_1 \cos(\beta x) + k_2 \sin(\beta x)]$ , for constants  $c_1, c_2, k_1, k_2$ .



## 3.3 Homogenous Linear Equations with Constant Coefficients:

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- 5. Solve the IVP  $y'' + 2y' = 0$ ,  $y(0) = 1$ ,  $y'(0) = 1$ .
- 6. Find a general solution for  $6y^{(4)} - y''' + 4y'' - y' - 2y = 0$ .

