Math 2Z03 - Tutorial # 3

Sept. 28th, 29th, 30th, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

Tutorial #3:

- 2.8 Nonlinear Models
  - Logistic Equation
- 2.6 A Numerical Method
  - Euler's Method
- 2.3 First-Order Linear Equations
  - Method of Solution
- 2.7 Linear Models
  - Mixture of Two Salt Solutions

Last Tutorial:

- We solved the logistic equation

$$\frac{dP}{dt} = P(a - bP), P(0) = P_0.$$

- We found the solution to be:

$$P = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

2.8 Nonlinear Models (Logistic Equation)

- **1.** Suppose that the population P (in thousands) of squirrels in Hamilton can be modelled by the differential equation $\frac{dP}{dt} = P(2-P)$, where t is the number of years.

  a) If the initial population of squirrels is 3000, what can you say about the long-term behaviour of the squirrel population?

  b) Can a population of 1000 ever decline to 500? Explain.

  c) Can a population of 1000 ever increase to 3000? Explain.

  d) If the initial population of squirrels in 50, then how many squirrels will there be after one year?

2.8 Nonlinear Models (Logistic Equation)

- To solve d) we need to solve the logistic equation with $a = 2$ and $b = 1$.
- **Remark:** Instead of memorizing the logistic equation, you could just compute the solution directly by separating variables and using partial fractions.

2.6 A Numerical Method (Euler's Method)

- **Recall:** Euler's Method approximates the solution to a first-order IVP $y' = f(x,y)$, $y(x_0) = y_0$ using the following recursive formula:

$$y_{n+1} = y_n + hf(x_n, y_n),$$

  where $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$.
  The **step size**, h, is given in advance and is chosen to be reasonably small.

- **2.** Consider the IVP $y' = 2x - 3y + 1$, $y(1) = 5$. Find an approximation of $y(1.2)$ using Euler's method with a step size of $h = 0.1$.

2.3 First-Order Linear Equations

- **Recall:** A **first-order linear DE** is an equation that can be written in the form

$$y' + P(x)y = f(x).$$

- **Method of Solution:**The solution to a first-order linear DE is given by:

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} f(x)dx\right].$$

2.3 First-Order Linear Equations

- **Note:** Instead of memorizing this formula, you may prefer to do the following:

  1. Identify $P(x)$ and compute the integrating factor $e^{\int P(x)dx}$.
  2. Multiply your linear equation $y' + P(x)y = f(x)$ by the integrating factor.
  3. The lefthand side of the resulting equation is automatically the derivative of the integrating factor and y. Write

  $$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x),$$

  and then integrate both sides of the equation.

2.3 First-Order Linear Equations

- **3.a**) Solve the differential equation

$$xy' - y = 2x\ln x.$$

- **b)** Find the largest interval where this solution is defined.

2.6 Linear Models: Mixtures

- **4.** ***(2.6,# 27)*** A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing $\frac{1}{2}$ pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

- Let $A(t)$ denote the amount of salt in the tank at time t (measured in lb).

- We know

$$\frac{dA}{dt} = \underbrace{(\text{input rate of salt})}_{R_{in}} - \underbrace{(\text{output rate of salt})}_{R_{out}}.$$

2.6 Linear Models: Mixtures

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- $R_{in} =$ (concentration of salt inflow lb/gal) $*$ (input rate of brine gal/min) $=$ (input rate lb/min).

- $R_{out} =$ (concentration of salt outflow lb/gal) $*$ (output rate of brine gal/min) $=$ (output rate lb/min).