Math 2Z03 - Tutorial # 3

Sept. 28th, 29th, 30th, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

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- 2.8 Nonlinear Models

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- 2.8 Nonlinear Models
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  - Euler's Method
- 2.3 First-Order Linear Equations
  - Method of Solution
- 2.7 Linear Models
  - Mixture of Two Salt Solutions

Last Tutorial:

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- We found the solution to be:

$$P = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

2.8 Nonlinear Models (Logistic Equation)

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  c) Can a population of 1000 ever increase to 3000? Explain.

  d) If the initial population of squirrels in 50, then how many squirrels will there be after one year?

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- **Remark:** Instead of memorizing the logistic equation, you could just compute the solution directly by separating variables and using partial fractions.

2.6 A Numerical Method (Euler's Method)

- **Recall:** Euler's Method approximates the solution to a first-order IVP $y' = f(x,y)$, $y(x_0) = y_0$ using the following recursive formula:

$$y_{n+1} = y_n + hf(x_n, y_n),$$

where $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$.
The **step size**, h, is given in advance and is chosen to be reasonably small.

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- **2.** Consider the IVP $y' = 2x - 3y + 1$, $y(1) = 5$. Find an approximation of $y(1.2)$ using Euler's method with a step size of $h = 0.1$.

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- **Method of Solution:** The solution to a first-order linear DE is given by:

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} f(x)dx\right].$$

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  1. Identify $P(x)$ and compute the integrating factor $e^{\int P(x)dx}$.
  2. Multiply your linear equation $y' + P(x)y = f(x)$ by the integrating factor.
  3. The lefthand side of the resulting equation is automatically the derivative of the integrating factor and y. Write

  $$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x),$$

  and then integrate both sides of the equation.

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- **b)** Find the largest interval where this solution is defined.

2.6 Linear Models: Mixtures

- **4.** ***(2.6,# 27)*** A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing $\frac{1}{2}$ pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

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- We know

$$\frac{dA}{dt} = \underbrace{(\text{input rate of salt})}_{R_{in}} - \underbrace{(\text{output rate of salt})}_{R_{out}}.$$

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