Math 2Z03 - Tutorial # 2

Sept. 21st, 22nd, 23rd, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

Tutorial #2:

- 2.2 Separable Equations
  - *Review Integration*
- 2.1 Solution Curves Without a Solution
  - Autonomous DE's
- 2.8 Nonlinear Models
  - Logistic Equation
- 2.6 A Numerical Method
  - Euler's Method

2.2: Separable Equations

- **Recall:** A **separable equation** is a first order DE which can be written in the form
$$y' = g(x)h(y).$$

- **Method of Solution:** Integrate $\int \frac{1}{h(y)}dy = \int g(x)dx$.
- **1.** Solve the following differential equations.
  a) $e^{-x^2}y' = e^{-y}x$.
  b) $\frac{1}{x}y' = y^{-1}e^{2x}$.
  c) $y' = e^{x^2}$.
- **Notice:** The functions xe^{x^2}, xe^{2x}, and e^{x^2} look very similar. However, to integrate, the first uses **substitution**, the second uses **integration by parts**, and the third isn't possible to integrate by hand!

2.1.2: Autonomous DE's

- **Recall:** A first order **autonomous** differential equation is a DE in which the independent variable does not appear explicitly. i.e. it can be written in the form
$$y' = f(y).$$

- We can analyze autonomous DE's geometrically (use phase portraits to sketch solution curves).

- This gives us information about the behaviour of solutions.

2.1.2: Autonomous DE's

- **2. a)** Sketch the solution curves of the DE $y' = y^2 - y^3$.
  - Find critical points.
  - Draw phase portrait.
  - Sketch solutions.

- **b)** Create an initial value problem (IVP) involving this DE, such that in the long term the solution to your IVP approaches zero.

2.8 Nonlinear Models (Logistic Equation)

- **Logistic Equation:** models simple population growth.
- Suppose a population grows proportional to its size:

$$\frac{dP}{dt} = kP, k > 0.$$

  - bacteria growth
  - simple investment with compound interest

- Solving this DE:

$$P = P_0 e^{kt} \textit{(exponential growth)}$$

2.8 Nonlinear Models (Logistic Equation)

- Similarly,

$$\frac{dP}{dt} = -kP, k > 0 \quad \textit{(exponential decay)}$$

  - radioactive decay
  - breakdown of a chemical

- In these two simple models, the growth is unbounded.

- Suppose an environment can sustain no more than $\bar{P}$ individuals in its population ($\bar{P}$ is called the **carrying capacity**).

- When $P > \bar{P}$ we want $\frac{dP}{dt} < 0$. The DE which models this is:

$$\frac{dP}{dt} = kP(\bar{P} - P).$$

2.8 Nonlinear Models (Logistic Equation)

- Making the substitution $\bar{P} = \frac{a}{b}$, $k = b$, this equation becomes:

$$\underbrace{\frac{dP}{dt} = P(a - bP)\,.}_{\textit{Logistic Equation}}$$

- **Solving the Logistic Equation:** It's autonomous, and therefore separable *(all autonomous DE's are separable).*

Review: Partial Fractions

- Consider a rational function $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials with real coefficients, and $\deg(P(s)) < \deg(Q(s))$.
  1. Factor and cancel common factors of $P(s)$ and $Q(s)$.
  2. For each linear term $(s-a)^m$, $a \in \mathbb{R}$, in the denominator, include terms of the form:
  $$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}.$$
  3. For each irreducible quadratic term $[(s-\alpha)^2 = \beta^2]^p$, $\alpha, \beta \in \mathbb{R}$, include terms of the form
  $$\frac{B_1 s + C_1}{(s-\alpha)^2 + \beta^2} + \frac{B_2 s + C_2}{((s-\alpha)^2 + \beta^2)^2} + \cdots + \frac{B_p s + C_p}{((s-\alpha^2)^p + \beta^2)^p}.$$

Review: Partial Fractions

4. Set $\frac{P(s)}{Q(s)}$ equal to the sum of these terms.
5. Put over common denominator.
6. Equate numerators.
7. a) Find A_i, B_i, C_i by equating coefficients s^k.
   b) Evaluate both sides at the roots.

2.8 Nonlinear Models (Logistic Equation)

- Making the substitution $\bar{P} = \frac{a}{b}$, $k = b$, this equation becomes:

$$\underbrace{\frac{dP}{dt} = P(a - bP)\,.}_{\textit{Logistic Equation}}$$

- **Solving the Logistic Equation:** It's autonomous, and therefore separable. *(all autonomous DE's are separable)*

- $$P = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

2.8 Nonlinear Models (Logistic Equation)

- **3.** Suppose that the population P (in thousands) of squirrels in Hamilton can be modelled by the differential equation $\frac{dP}{dt} = P(2-P)$.

  a) If the initial population of squirrels is 3000, what can you say about the long-term behaviour of the squirrel population?

  b) Can a population of 1000 ever decline to 500? Explain.

  c) Can a population of 1000 ever increase to 3000? Explain.

- This question can be answered without explicitly solving for the solution, P. However, if we wanted to write down the explicit solution, using the logistic equation with $a = 2$ and $b = 1$ we obtain:

$$P = \frac{2P_0}{P_0 + (2-P_0)e^{-2t}}.$$

2.6 A Numerical Method (Euler's Method)

- **Recall:** Euler's Method approximates the solution to a first-order IVP $y' = f(x, y)$, $y(x_0) = y_0$ using the following recursive formula:

  $$y_{n+1} = y_n + hf(x_n, y_n),$$

  where $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$.
  The **step size**, h, is given in advance and is chosen to be reasonably small.

- **4.** Consider the IVP $y' = 2x - 3y + 1$, $y(1) = 5$. Find an approximation of $y(1.2)$ using Euler's method with a step size of $h = 0.1$.