Math 2Z03 - Tutorial # 2

Sept. 21st, 22nd, 23rd, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

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- **Notice:** The functions xe^{x^2}, xe^{2x}, and e^{x^2} look very similar. However, to integrate, the first uses **substitution**, the second uses **integration by parts**, and the third isn't possible to integrate by hand!

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- This gives us information about the behaviour of solutions.

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- **b)** Create an initial value problem (IVP) involving this DE, such that in the long term the solution to your IVP approaches zero.

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- Solving this DE:

$$P = P_0 e^{kt} \textit{(exponential growth)}$$

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- Suppose an environment can sustain no more than $\bar{P}$ individuals in its population ($\bar{P}$ is called the **carrying capacity**).
- When $P > \bar{P}$ we want $\frac{dP}{dt} < 0$. The DE which models this is:

$$\frac{dP}{dt} = kP(\bar{P} - P).$$

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- **Solving the Logistic Equation:** It's autonomous, and therefore separable *(all autonomous DE's are separable).*

Review: Partial Fractions

- Consider a rational function $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials with real coefficients, and $\deg(P(s)) < \deg(Q(s))$.

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  1. Factor and cancel common factors of $P(s)$ and $Q(s)$.
  2. For each linear term $(s-a)^m$, $a \in \mathbb{R}$, in the denominator, include terms of the form:
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  3. For each irreducible quadratic term $[(s-\alpha)^2 = \beta^2]^p$, $\alpha, \beta \in \mathbb{R}$, include terms of the form
  $$\frac{B_1 s + C_1}{(s-\alpha)^2 + \beta^2} + \frac{B_2 s + C_2}{((s-\alpha)^2 + \beta^2)^2} + \cdots + \frac{B_p s + C_p}{((s-\alpha^2)^p + \beta^2)^p}.$$

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7. a) Find A_i, B_i, C_i by equating coefficients s^k.
   b) Evaluate both sides at the roots.

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- $$P = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

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- **3.** Suppose that the population P (in thousands) of squirrels in Hamilton can be modelled by the differential equation $\frac{dP}{dt} = P(2 - P)$.

  a) If the initial population of squirrels is 3000, what can you say about the long-term behaviour of the squirrel population?

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- This question can be answered without explicitly solving for the solution, P. However, if we wanted to write down the explicit solution, using the logistic equation with $a = 2$ and $b = 1$ we obtain:

$$P = \frac{2P_0}{P_0 + (2-P_0)e^{-2t}}.$$

2.6 A Numerical Method (Euler's Method)

- **Recall:** Euler's Method approximates the solution to a first-order IVP $y' = f(x,y)$, $y(x_0) = y_0$ using the following recursive formula:

$$y_{n+1} = y_n + hf(x_n, y_n),$$

where $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$.
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- **4.** Consider the IVP $y' = 2x - 3y + 1$, $y(1) = 5$. Find an approximation of $y(1.2)$ using Euler's method with a step size of $h = 0.1$.