Math 2Z03 - Tutorial #12

Dec. 7th, 8th, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html

Tutorial #12**:**

- 5.1: Series Solutions of Linear DE's about Ordinary Points
- Chapter 4: The Laplace Transform

5.1: Series Solutions of Linear DE's about Ordinary Points

- **1.** Find the minimum radius convergence of the given DE about the given ordinary point. **a)** $(x^2 - 25)y'' + 2xy' + y = 0$, $x = 1$.

- **Existence of Power Series Solutions:** If $x = x_0$ is an *ordinary point* of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then we can always find 2 linearly independent power series solutions centered at x_0 (i.e. of the form $\sum_{n=0}^{\infty} c_n(x - x_0)^n$), with **minimum radius of convergence** R, where R is the distance from x_0 to the nearest *singular point*.

- **Fact:** If $a_2(x)$, $a_1(x)$, and $a_0(x)$ are polynomials with no common factors, then the point x_0 is an **ordinary point** of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ if $a_2(x) \neq 0$. If $a_2(x_0) = 0$, then x_0 is a **singular point**.

5.1: Series Solutions of Linear DE's about Ordinary Points

- **Recall:** A point x_0 is said to be an **ordinary point** of $y'' + P(x)y' + Q(x)y = 0$ if both $P(x)$ and $Q(x)$ are *analytic* at x_0. A point that is not an ordinary point is called a **singular point**.
- **Recall:** A function f is **analytic** at a point x_0 if it can be represented by a power series in $x - x_0$ with a positive radius of convergence.
- **b)** Find the minimum radius convergence of the given DE about the given ordinary point: $(x^2 - 2x + 10)y'' + xy' - 4y = 0$, $x = 1$.

5.1: Series Solutions of Linear DE's about Ordinary Points

- **2. a)** Find the first 5 nonzero terms in the general solution of $(1+x^2)y'' - y' + y = 0$ (about the ordinary point $x = 0$).
- **Method of Solution:**
  1. Substitute $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ into the original DE (**i.e.** find y' and y'', and bring any coefficients in front of the sum into the sum).
  2. Combine the series. **i.e.** In each series, take the exponent of the x term and set it equal to k, so that after the substitution, each series has x^k.
  3. Equate all coefficients to the RHS of the equation to determine the coefficients c_n and the recurrence relation.
  4. Write all coefficients in terms of c_0 and c_1. We can always choose values for c_0 and c_1 to identify two linearly independent solutions.
- **b)** Explain why this is the general solution.(**i.e.** Identify 2 linearly independent solutions).

5.1: Series Solutions of Linear DE's about Ordinary Points

- **3. a)** Find the first 4 nonzero terms of a general solution to $2y'' + xy' + y = 0$ (about the ordinary point $x = 0$).
- **b)** Find the first 2 nonzero terms of a solution (about $x = 0$) of the IVP $2y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Chapter 4: The Laplace Transform

- **(Tutorial # 11, 8.)** Solve $y' - 3y = \delta(t-2), y(0) = 0$.
- **Recall:** Find $\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$.
- **(Tutorial # 11, 9.)** Use the Laplace transform to solve the system
$$\begin{cases} x' = -x + y \\ y' = 2x \\ x(0) = 0, y(0) = 1 \end{cases}$$

Helpful Formulas. **DO NOT DETACH!**

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_0^\infty f(t)e^{-st}dt,$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}, \qquad \mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}, \qquad \mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2},$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-sa}F(s) \qquad \mathcal{L}\{\mathcal{U}(t-a)f(t)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}\{t^nf(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathcal{L}\{f*g\} = F(s)G(s) \qquad f*g = \int_0^t f(t-v)g(v)dv$$

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B),$$
$$2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B),$$
$$2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B),$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B),$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$