Math 2Z03 - Tutorial #11

Nov. 30th, Dec. 1st, Dec. 2nd, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

Tutorial #11**:**

- Chapter 4: The Laplace Transform

Chapter 4: The Laplace Transform

- **1.** Use the Laplace transform to solve the IVP

$$y'' + y = \sqrt{2}\sin(\sqrt{2}t), y(0) = 10, y'(0) = 0.$$

Review: Partial Fractions

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  3. For each irreducible quadratic term $[(s-\alpha)^2 = \beta^2]^p$, $\alpha, \beta \in \mathbb{R}$, include terms of the form
  $$\frac{B_1 s + C_1}{(s-\alpha)^2 + \beta^2} + \frac{B_2 s + C_2}{((s-\alpha)^2 + \beta^2)^2} + \cdots + \frac{B_p s + C_p}{((s-\alpha^2)^p + \beta^2)^p}.$$

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7. a) Find A_i, B_i, C_i by equating coefficients s^k.
   b) Evaluate both sides at the roots.

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- **5.** Find $\mathscr{L}^{-1}\left\{\frac{6}{(s-5)^4}.\right\}$

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  - The **convolution** of f and g is: $f * g = \int_0^t f(\tau)g(t-\tau)d\tau$.
  - $\mathscr{L}\{f * g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$.

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- **8.** Solve $y' - 3y = \delta(t-2), y(0) = 0$.

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- **8.** Solve $y' - 3y = \delta(t-2), y(0) = 0$.
- **Recall:** Find $\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$.

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- **9.** Use the Laplace transform to solve the system

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$$\begin{cases} x' = -x + y \\ y' = 2x \\ x(0) = 0, y(0) = 1 \end{cases}$$

- **10.** Find $\mathscr{L}\left\{te^{2t}\sin(6t)\right\}$.

Helpful Formulas. **DO NOT DETACH!**

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_0^\infty f(t)e^{-st}dt,$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}, \qquad \mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}, \qquad \mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2},$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-sa}F(s) \qquad \mathcal{L}\{\mathcal{U}(t-a)f(t)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}\{t^nf(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathcal{L}\{f*g\} = F(s)G(s) \qquad f*g = \int_0^t f(t-v)g(v)dv$$

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B),$$
$$2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B),$$
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