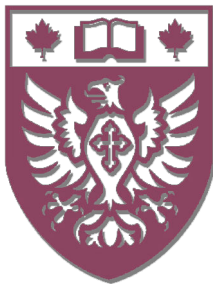


Math 2Z03 - Tutorial #11



Nov. 30th, Dec. 1st, Dec. 2nd,
2015

Tutorial Info:

- **Tutorial Website:** <http://ms.mcmaster.ca/~dedieula/2Z03.html>
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)



Tutorial #11:

- Chapter 4: The Laplace Transform



Chapter 4: The Laplace Transform

- 1. Use the Laplace transform to solve the IVP

$$y'' + y = \sqrt{2}\sin(\sqrt{2}t), y(0) = 10, y'(0) = 0.$$



Review: Partial Fractions

- Consider a rational function $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials with real coefficients, and $\deg(P(s)) < \deg(Q(s))$.



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 2. For each linear term $(s - a)^m$, $a \in \mathbb{R}$, in the denominator, include terms of the form:

$$\frac{A_1}{s - a} + \frac{A_2}{(s - a)^2} + \cdots + \frac{A_m}{(s - a)^m}.$$



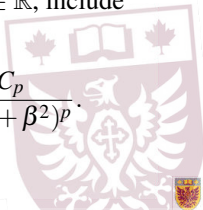
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3. For each irreducible quadratic term $[(s - \alpha)^2 + \beta^2]^p$, $\alpha, \beta \in \mathbb{R}$, include terms of the form

$$\frac{B_1s + C_1}{(s - \alpha)^2 + \beta^2} + \frac{B_2s + C_2}{((s - \alpha)^2 + \beta^2)^2} + \cdots + \frac{B_ps + C_p}{((s - \alpha)^2 + \beta^2)^p}.$$



Review: Partial Fractions

4. Set $\frac{P(s)}{Q(s)}$ equal to the sum of these terms.



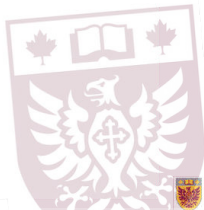
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b) Evaluate both sides at the roots.



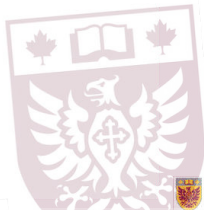
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- 5. Find $\mathcal{L}^{-1}\left\{\frac{6}{(s-5)^4}\right\}$



Chapter 4: The Laplace Transform

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- **Recall:**
 - The **convolution** of f and g is: $f * g = \int_0^t f(\tau)g(t - \tau)d\tau$.
 - $\mathcal{L} \{f * g\} = \mathcal{L} \{f\} \mathcal{L} \{g\}$.



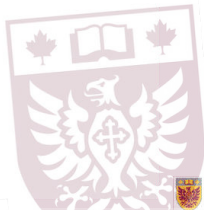
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- 8. Solve $y' - 3y = \delta(t - 2), y(0) = 0$.



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- 8. Solve $y' - 3y = \delta(t - 2), y(0) = 0$.
- **Recall:** Find $\mathcal{L} \{\delta(t - t_0)\} = e^{-st_0}$.



Chapter 4: The Laplace Transform

- 9. Use the Laplace transform to solve the system

$$\begin{cases} x' = -x + y \\ y' = 2x \\ x(0) = 0, y(0) = 1 \end{cases}$$



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- 10. Find $\mathcal{L} \{te^{2t} \sin(6t)\}$.



Helpful Formulas. DO NOT DETACH!

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st} dt,$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}, \quad \mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2},$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2 - k^2}, \quad \mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2 - k^2},$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-sa}F(s) \quad \mathcal{L}\{\mathcal{U}(t-a)f(t)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}\{f * g\} = F(s)G(s) \quad f * g = \int_0^t f(t-v)g(v)dv$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$2 \sin(A) \sin(B) = \cos(A-B) - \cos(A+B),$$

$$2 \sin(A) \cos(B) = \sin(A+B) + \sin(A-B),$$

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$$\sin(A+B) = \sin(A) \cos(B) + \cos(A) \sin(B),$$

$$\cos(A+B) = \cos(A) \cos(B) - \sin(A) \sin(B).$$