Math 2Z03 - Tutorial #10

Nov. 23rd, 24th, 25th, 2015

Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/∼dedieula/2Z03.html
- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)

Tutorial #8:

- 10.2 Homogeneous Linear Systems
- 4.1 Definition of the Laplace Transform
- 4.2 The Inverse Transform and Transforms of Derivatives

10.2 Homogeneous Linear Systems

- **1. a)** Find the general solution for the homogeneous system of linear DE's:

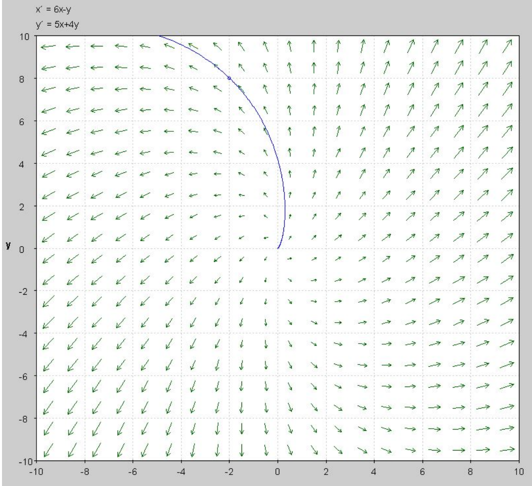
$$X' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} X, X(0) = \left[\begin{smallmatrix} -2 \\ 8 \end{smallmatrix}\right].$$

- **Recall:** Consider the homogeneous linear DE $X' = AX$. If $\lambda = \alpha + \beta i$ is an eigenvalue of the coefficient matrix A, with corresponding eigenvector $v = B_1 + B_2 i$, then two linearly independent solutions of this system on $(-\infty, \infty)$ are:

$$X_1 = e^{\alpha t}\left[B_1 \cos(\beta t) - B_2 \sin(\beta t)\right]$$
$$X_2 = e^{\alpha t}\left[B_1 \sin(\beta t) + B_2 \cos(\beta t)\right].$$

- **b)** Sketch the solution curve corresponding to this IVP.

10.2 Homogeneous Linear Systems



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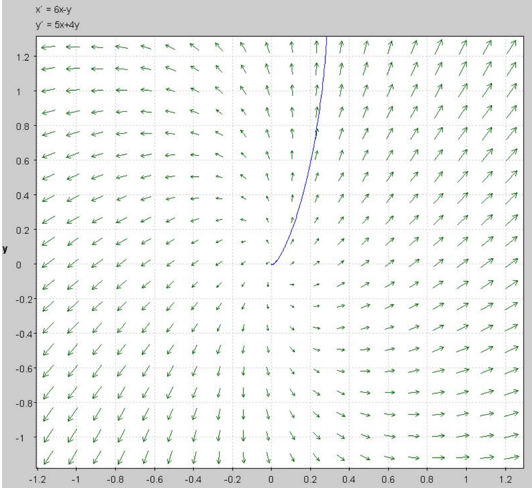

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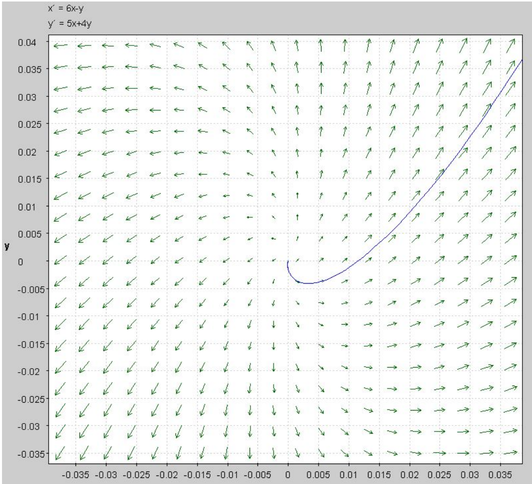

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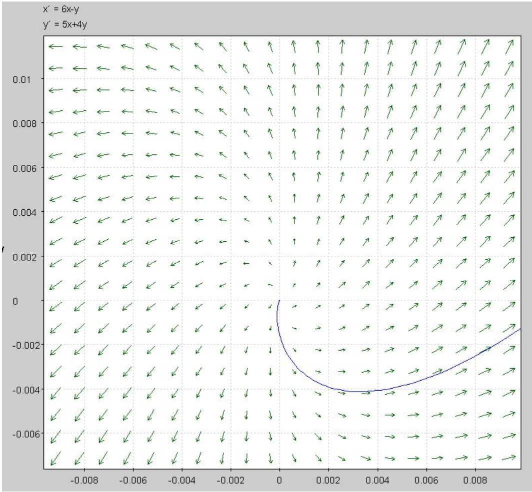

10.2 Homogeneous Linear Systems

- **2.** Find the general solution for

$$X' = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} X.$$

10.2 Homogeneous Linear Systems

- **Recall:** Consider the homogeneous linear system $X' = AX$. If A has an eigenvalue λ of multiplicity m with only one corresponding eigenvector K, then you can always find n linearly independent solutions of the form

$$X_1 = Ke^{\lambda t}$$
$$X_2 = (Kt + P_1)e^{\lambda t}$$
$$X_3 = (\frac{t^2}{2}K + P_1 t + P_2)e^{\lambda t}, \ldots \text{ etc.}$$

$$X_1 \text{solution } \Rightarrow (A - \lambda I)K = 0$$
$$X_2 \text{ solution } \Rightarrow (A - \lambda I)K = 0 \text{ and } (A - \lambda I)P_1 = K$$
$$X_3 \text{ solution } \Rightarrow (A - \lambda I)K = 0 \text{ and } (A - \lambda I)P_1 = K \text{ and } (A - \lambda I)P_2 = P_1$$

4.1/4.2 The Laplace Transform

- **3.** Find $\mathscr{L}\left\{(1+e^{2t})^2\right\}$.
- **4.** Find $\mathscr{L}\{t\}$ using the formal definition of the Laplace transform.
- **Recall:** $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$.
- **5.** Find $\mathscr{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$.

Review: Partial Fractions

- Consider a rational function $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials with real coefficients, and $\deg(P(s)) < \deg(Q(s))$.

  1. Factor and cancel common factors of $P(s)$ and $Q(s)$.
  2. For each linear term $(s-a)^m$, $a \in \mathbb{R}$, in the denominator, include terms of the form:
  $$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}.$$
  3. For each irreducible quadratic term $[(s-\alpha)^2 = \beta^2]^p$, $\alpha, \beta \in \mathbb{R}$, include terms of the form
  $$\frac{B_1 s + C_1}{(s-\alpha)^2 + \beta^2} + \frac{B_2 s + C_2}{((s-\alpha)^2 + \beta^2)^2} + \cdots + \frac{B_p s + C_p}{((s-\alpha^2)^p + \beta^2)^p}.$$

Review: Partial Fractions

4. Set $\frac{P(s)}{Q(s)}$ equal to the sum of these terms.
5. Put over common denominator.
6. Equate numerators.
7. a) Find A_i, B_i, C_i by equating coefficients s^k.
   b) Evaluate both sides at the roots.

4.1/4.2 The Laplace Transform

- **6.** Use the Laplace transform to solve the linear IVP

$$2y' + y = 0, y(0) = -3.$$

- **Recall:** To solve this we want to $\mathscr{L}$ both sides, isolate for $\mathscr{L}\{y\} := Y(s)$, then $\mathscr{L}^{-1}$ both sides.

Table of Laplace Transforms:

Here $\mathscr{L}\{f(t)\} = F(s)$.

Transforms of Some Basic Functions

- $\mathscr{L}\{1\} = \dfrac{1}{s}$
- $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$
- $\mathscr{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}$
- $\mathscr{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$
- $\mathscr{L}\{\sinh(kt)\} = \dfrac{k}{s^2-k^2}$
- $\mathscr{L}\{\cosh(kt)\} = \dfrac{s}{s^2-k^2}$

Transforms of Derivatives

- $\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$

Translation Theorems

- $\mathscr{L}\{e^{at} f(t)\} = \mathscr{L}\{f(t)\} \mid_{s \to s-a} = F(s-a)$, where $a \in \mathbb{R}$
- $\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s) \mid_{s \to s-a}\} = e^{at} f(t)$
- $\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as} F(s)$, where $a > 0$
- $\mathscr{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\mathscr{U}(t-a)$, where $a > 0$
- $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$, where $a > 0$
- $\mathscr{L}\{\mathscr{U}(t-a)\} = \dfrac{e^{-as}}{s}$, where $a > 0$

Derivatives of Transforms & Convolution

- $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s), n = 1, 2, \ldots$
- $\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$

Dirac Delta Function

- $\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$, for $t_0 > 0$