MATHEMATICS 2C03 PRACTICE FINAL EXAM

DAY SECTIONS 01, 02
DURATION of FINAL EXAM: 3 HOURS
McMASTER UNIVERSITY

THIS EXAMINATION INCLUDES 20 PAGES AND 20 QUESTIONS. IT IS POSSIBLE TO OBTAIN A TOTAL OF 102 MARKS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR. A **Table of Laplace Transforms** is provided on the top of page 3.

SPECIAL INSTRUCTIONS:

Page 20 is to be used for continuation of a problem if you run out of space. **Ask the invigilators** for **SCRAP paper** if required. **NO NOTES OR AIDS OR PIECES OF PAPER OF ANY KIND (other than that distributed by the invigilator) ARE PERMITTED.**

You are **NOT permitted** to have any **ELECTRONIC DEVICES** of any kind, including calculators and cell phones.

PART I is made up of **14 Multiple choice questions**. **MARK YOUR ANSWERS ON THE OMR EXAMINATION SHEET with HB pencil ONLY.** For this part, only the OMR Examination sheet will be marked. Each multiple choice question is worth **3 marks**. There is **no penalty** for an incorrect answer.

PART II is made up of **5 Complete Answer Questions**. Each question is of equal value.

You must print your name and ID number at the **top of each complete answer page** in the space provided **as well as** on this page below. **You MUST hand in both the OMR EXAMINATION SHEET** and **this examination paper.**

NAME: * Solutions *

ID #: ____________ Tutorial #: ____________

Questions	Mark	Out of
15		12
16		12
17		12
18		12
19		12
TOTAL		60

GOOD LUCK!

continued ...

Section 7.4:

- $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$, $n = 1, 2, \ldots$
- $\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$

Section 7.5:

- $\mathscr{L}\{\delta(t - t_0)\} = e^{-st_0}$, for $t_0 > 0$

PART I: MULTIPLE CHOICE

There are 14 multiple choice questions. All questions have the same value. A correct answer scores 3 and an incorrect answer scores zero. Record your answer on the OMR Examination Answer Sheet provided. Follow the instructions on page 2 carefully. **Use HB pencil only.** Ask the invigilators for scrap paper if required.

1. The differential equation $\underbrace{(2xy + x^2 + x^4)}_{M}dx + \underbrace{(1 + x^2)}_{N}dy = 0$, is:

   (A) homogeneous and exact. ✗

   (B) separable and homogeneous. ✗

   (C) separable and exact.

   (D) linear and homogeneous. ✗

   (E) linear and exact.

• Not homog. since $N(tx, ty) = 1 + t^2x^2 \neq t^4 N(x,y)$.

• $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \Rightarrow$ exact.

• $(1+x^2)y' = -2xy - x^2 - x^4$

$\Rightarrow (1+x^2)y' + 2xy = -x^2 - x^4 \Rightarrow$ linear.

• Not separable... no way to get the "y" out of M.

$\therefore$ E.

continued ...

2. An integrating factor of the form $\mu(x)$ for $\underbrace{(2x^2+y)}_{M}dx + \underbrace{x(xy-1)}_{N}dy = 0$ is:

(A) x^2
(B) x^{-1}
(C) x^{-2}
(D) e^x
(E) xe^x

$\frac{\partial M}{\partial y} = 1.$ $\frac{\partial N}{\partial x} = 2xy - 1.$

$\frac{M_y - N_x}{N} = \frac{1-2xy+1}{x^2y - x} = \frac{2-2xy}{x(xy-1)}$

$= \frac{-2(-1+xy)}{x(xy-1)} = -2/x.$

$\therefore \mu(x) = e^{\int -2/x\, dx} = e^{-2\ln x} = x^{-2}.$

Check:

$\underbrace{(2+x^{-2}y)}_{\tilde{M}}dx + \underbrace{(y-x^{-1})}_{\tilde{N}}dy = 0$

$\frac{\partial \tilde{M}}{\partial y} = x^{-2} = \frac{\partial \tilde{N}}{\partial x}.$ ✓

3. If $y(x)$ is the solution of the initial value problem,

$x^2y'' - 3xy' + 4y = 0,\ y(1) = 0,\ y'(1) = 2,$ then $y(e)$ equals:

(A) $2e^2$
(B) e^2+1
(C) $2(e^2+1)$
(D) e^2+2e
(E) e^2+2

Cauchy-Euler.

$x^k\,[k(k-1) - 3k + 4] = 0$

$k^2 - 4k + 4$

$(k-2)^2 \Rightarrow k = 2$ double root.

$y = c_1x^2 + (\ln x)x^2c_2.$

$y' = 2c_1x + c_2x + 2(\ln x)xc_2.$

$y(1) = 0 \Rightarrow 0 = c_1.$

$y'(1) = 2 \Rightarrow 2 = 2c_1 + c_2 = c_2.$

$\therefore y = 2x^2\ln x.$

So, $y(e) = 2e^2.$

continued ...

4. If $y(x)$ is the solution of the initial value problem,

$y' + \underbrace{\frac{1}{x}}_{P(x)}y = \underbrace{x^2}_{F(x)}y^2$, $y(1) = 2$, then $y(2)$ equals:

(A) $-\frac{3}{2}$

(B) $-\frac{1}{2}$

(C) $-\frac{1}{4}$

(D) 0

(E) $\frac{2}{3}$

Bernoulli's eq'n w/ $n=2$.

$u = y^{1-n} = y^{-1}$

$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$

$-y^2 u' = y'$.

$x^2 = \underbrace{-y^{-2}y'}_{u'} + \frac{1}{x}\underbrace{y^{-1}}_{u}$

$u' - \frac{1}{x}u = -x^2$ linear in u.

$\int -\frac{1}{x}\,dx = -\ln x$.

$u = e^{\ln x}\int e^{-\ln x}(-x^2)\,dx = -x\int x\,dx = -x\left[\frac{1}{2}x^2 + c\right]$.

So, $y = \left(-\frac{1}{2}x^3 + cx\right)^{-1}$, so $y(1)=2$

$\Rightarrow 2 = \left(-\frac{1}{2} + c\right)^{-1} \Rightarrow \frac{1}{2} + \frac{1}{2} = c$

$\Rightarrow c = 1$.

$\Rightarrow y = \left(-\frac{1}{2}x^3 + x\right)^{-1}$ $\Rightarrow y(2) = (-2)^{-1} = -\frac{1}{2}$.

5. Find the general solution of the fourth-order homogeneous linear differential equation

$$y^{(4)} + 18y'' + 81y = 0.$$

(A) $y(x) = (c_1 + c_2x)\cos(3x) + (c_3 + c_4x)\sin(3x)$.

(B) $y(x) = c_1\cos(3x) + c_2\sin(3x)$.

(C) $y(x) = c_1 + c_2x + c_3\cos(3x) + c_4\sin(3x)$.

(D) $y(x) = c_1e^{3x} + c_2xe^{3x} + c_3\cos(3x) + c_4\sin(3x)$.

(E) $y(x) = (c_1x + c_2x^2)\cos(3x) + (c_3x + c_4x^2)\sin(3x)$.

$m^4 + 18m^2 + 81 = 0$

$(m^2+9)^2$

$\Rightarrow m^2 = -9$ double

$\Rightarrow m = \pm 3i$ double root.

$\alpha = 0, \beta = 3$.

$\therefore y_c = c_1\cos(3x) + c_2\sin(3x)$

$+ c_3 x\cos(3x) + c_4 x\sin(3x)$.

continued ...

6. Find the Laplace transform of $(t^2+3)u(t-2)$, where $u(t)$ is the unit step function.

(A) $e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{7}{s}\right)$
(B) $e^{-2s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{5}{s}\right)$
(C) $e^{-2s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{3}{s}\right)$
(D) $e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{9}{s}\right)$
(E) $e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s}\right)$

$\mathcal{L}\{\underbrace{(t^2+3)}_{g(t)} u(t-2)\} = e^{-2s}\mathcal{L}\{(t+2)^2+3\}$
$= e^{-2s}\mathcal{L}\{t^2+4t+7\}$
$= e^{-2s}\left[\frac{2}{s^3}+\frac{4}{s^2}+\frac{7}{s}\right].$

7. Let $F(s)$ denote the Laplace transform of $f(t)=\int_0^t \underbrace{e^{-\tau}\cos(\tau)}_{(e^{-t}\cos t)*1}\,d\tau$.

Then, $F(2)$ is equal to:

(A) $\frac{1}{10}$
(B) $\frac{3}{20}$
(C) $\frac{1}{2}$
(D) $-\frac{1}{e}$
(E) 0

$F(s)=\mathcal{L}\{f(t)\}=\mathcal{L}\{e^{-t}\cos t * 1\}$
$=\mathcal{L}\{e^{-t}\cos t\}\,\mathcal{L}\{1\} = \frac{1}{s}\mathcal{L}\{\cos t\}\Big|_{s\to s+1}$
$=\frac{1}{s}\,\frac{(s+1)}{(s+1)^2+1} \Rightarrow F(2)=\frac{1}{2}\,\frac{3}{10}=\frac{3}{20}.$

8. The inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+11}\right\}$ is:

(A) $\cos(\sqrt{2}(t-3))-\frac{3}{2}\sin(\sqrt{2}(t-3))$
(B) $e^{-3t}(\cos(\sqrt{2}\,t)-\frac{3}{\sqrt{2}}\sin(\sqrt{2}\,t))$
(C) $t\cos(\sqrt{2}\,t)\,u(t-\sqrt{2})$
(D) $e^{-3t}(\cos(\sqrt{2}\,t)-\sin(\sqrt{2}\,t))$
(E) $e^{-3t}\cos(\sqrt{2}\,t)$

$\mathcal{L}^{-1}\left\{\frac{s+3-3}{(s+3)^2+2}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}\Big|_{s\to s+3}-3\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+2}\right\}=\cos(\sqrt{2}t)\,e^{-3t}-\frac{3}{\sqrt{2}}e^{-3t}\sin(\sqrt{2}t).$

continued ...

9. Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$, if $F(s) = \frac{s^2}{(s+1)(s^2+4)}$.

(A) $e^{-t} + \sin(2t)$

(B) $\frac{1}{2}(e^{-t} + 3\cos(2t) - 3\sin(2t))$

(C) $\frac{1}{4}(e^{-t} + 2\cos(2t) + 3\sin(2t))$

(D) $\frac{1}{5}(e^{-t} + 4\cos(2t) - 2\sin(2t))$

(E) $e^{-t} + 2\cos(2t) - 2\sin(2t)$

$$\frac{s^2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow s^2 = (s^2+4)A + (Bs+C)(s+1)$$

$$\Rightarrow 1 = 5A \Rightarrow A = \tfrac{1}{5}$$

$$\Rightarrow s^2 = s^2\left(\tfrac{1}{5} + B\right) + s(B+C) + \left(\tfrac{4}{5} + C\right)$$

$$\Rightarrow B = 1 - \tfrac{1}{5} = \tfrac{4}{5} \quad \& \quad C = -\tfrac{4}{5}$$

$$\mathcal{L}^{-1}\{F(s)\} = \tfrac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \tfrac{4}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \tfrac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \tfrac{1}{5}e^{-t} + \tfrac{4}{5}\cos(2t) - \tfrac{2}{5}\sin(2t).$$

10. The recursive formula for the coefficients a_n of the power series solution about $x_0 = 0$ of the differential equation $y'' - xy = 0$, is given by:

(A) $a_{n+2} = \frac{2na_{n-1}}{(n+2)(n+1)},\ n \geq 1.$

(B) $a_{n+2} = \frac{2(n-1)a_{n-1}}{n(n+1)},\ n \geq 1.$

(C) $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)},\ n \geq 1.$

(D) $a_{n+2} = \frac{na_{n-1}}{(n+1)},\ n \geq 1.$

(E) $a_{n+2} = \frac{a_{n-1}}{(n)(n+1)},\ n \geq 1.$

$$0 = \underbrace{\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}}_{K=n-2} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{K=n+1}$$

$$= \sum_{K=0}^{\infty} c_{K+2}(K+2)(K+1)x^K - \sum_{K=1}^{\infty} c_{K-1}x^K$$

$$= 2c_2 + \sum_{K=1}^{\infty}\left[c_{K+2}(K+2)(K+1) - c_{K-1}\right]x^K$$

$$\Rightarrow c_{K+2} = \frac{c_{K-1}}{(K+2)(K+1)},\ K \geq 1.$$

continued ...

11. Which one of the functions listed below is a power series solution about the ordinary point $x_0 = 0$, of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, with radius of convergence $R = \infty$, for the differential equation $(1 + 2x^2)y'' + A(x)y' + B(x)y = 0$, given only that $A(x)$ and $B(x)$ are analytic functions, the coefficients a_0 and a_1 are arbitrary, and the recurrence formula is:

$$a_n = \frac{-(2n-5)(n-5)}{n(n-1)} a_{n-2}, \quad n \geq 2.$$

$a_0, a_1,$

$a_2 = \frac{-(-1)(-3)}{2} = -\frac{3}{2} a_0.$

$a_3 = \frac{-(1)(-2)}{3(2)} = \frac{2a_1}{6} = \frac{1}{3} a_1.$

$a_4 = -\frac{(3)(-1)}{4(3)} = \frac{3}{12} a_2 = \frac{1}{4}\left(\frac{-3}{2}\right.$

$= -\frac{3}{8} a_0.$

$a_5 = 0$ all odd zero from now on.

$a_6 = -\frac{(7)}{6 \cdot 5} a_4 = \frac{21}{8 \cdot 6 \cdot 5} a_0.$

(A) $3x + x^3$
(B) $2 + \frac{1}{3}x^3$
(C) $3x + \frac{1}{3}x^3 + \frac{2}{20}x^5$
(D) $1 + \frac{1}{2}x^2$
(E) $\sum_{k=1}^{\infty} (-1)^k \frac{[(-1)(3)(7)\cdots(4k-5)]x^{2k}}{k!\,(2k-3)(2k-1)}$

Choose $a_0 = 0$ & $a_1 = 3$:

$3x + \frac{1}{3}(3)x^3 + 0 + \ldots + 0 + \cdots$

$= 3x + x^3$. $R = \infty$.

12. The *best* one can say, without actually solving the differential equation, $(x^2 + 6x + 8)y'' + 6(x - 3)y' + 4y = 0$, is that the radius of convergence of the power series solution about, $x_0 = 3$, is *at least*:

(A) ∞
(B) 6
(C) 5
(D) 4
(E) 3

$P(x) = \frac{(x-3)}{(x^2+6x+8)} = \frac{(x-3)}{(x+4)(x+2)} \Rightarrow x = -2$ & $x = -4$ singular pts.

$Q(x) = \frac{4}{(x+4)(x+2)}$. Closest singular pt to $x_0 = 3$ is -2.

$3 - (-2) = 5.$

$\therefore$ min. radius of convergence is 5.

continued ...

13. A complete list of the *regular singular points* of the differential equation

$$x(x+2)^2(x-5)^2(x-1)^4y'' + (x-5)^2(x+3)(x-1)^3y' + (x-1)^2(x+2)y = 0$$

is given by:

(A) 1, 5
(B) 0, 1
(C) 0, 1, 5
(D) 0, 1, −2, 5
(E) 0, 1, −2, −3, 5

$$P(x) = \frac{(x-5)^2(x+3)(x-1)^3}{x(x+2)^2(x-5)^2(x-1)^4} = \frac{(x+3)}{x(x+2)^2(x-1)}$$

$\Rightarrow x=-2$ irregular.

$$Q(x) = \frac{(x-1)^2(x+2)}{x(x+2)^2(x-5)^2(x-1)^4} = \frac{1}{x(x+2)(x-5)^2(x-1)^2}$$

$\therefore$ Regular singular pts are $x = 0, 1, \& 5.$

14. The roots of the indicial equation for the point $x_0 = 0$ of the differential equation $x^2y'' + xy' + (x^2-1)y = 0$ are:

(A) $-1, 0$
(B) $-1, \frac{1}{2}$
(C) $\frac{1}{2}, \frac{3}{2}$
(D) $-1, 1$
(E) $0, \frac{1}{2}$

$P(x) = \frac{1}{x}$. So, $a_0 = \lim_{x\to 0} xP(x) = 1.$

$Q(x) = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2}$. So, $b_0 = \lim_{x\to 0} x^2 Q(x)$
$= \lim_{x\to 0} x^2 - 1 = -1.$

$r(r-1) + a_0 r + b_0 = 0$

$\Leftrightarrow r^2 - r + r - 1 = 0$

$\Leftrightarrow r^2 = 1$

$\Leftrightarrow r = \pm 1.$

continued ...

PART II: COMPLETE ANSWER QUESTIONS

Questions 15-19 Each question is of equal value.

You will be graded on the clarity and presentation of the solution, not just upon whether or not you obtain the correct answer. Scrap paper will NOT be graded. There are sheets at the end of this examnation for continuing solutions. If you need to use them, indicate that your are continuing the problem, and then indicate which problem you are continuing one the pagges at the end of this paper.

15. [12 marks] Find an explicit solution to the following differential equation:

$$\underbrace{(1 + e^x y + xe^x y)}_{M}dx + \underbrace{(xe^x + 2)}_{N}dy = 0.$$

$\left.\begin{array}{l}\frac{\partial M}{\partial y} = e^x + xe^x.\\ \frac{\partial N}{\partial x} = e^x + xe^x.\end{array}\right\} \Rightarrow$ exact $\Rightarrow$ $\frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N.$

$u = x$ $\quad v = e^x$
$du = dx$ $\quad dv = e^x dx$

$\frac{\partial F}{\partial x} = M \Rightarrow F = \int 1 + e^x y + xe^x y \, dx = x + e^x y + y\left[xe^x - \int e^x dx\right]$

$= x + \cancel{e^x y} + yxe^x - \cancel{e^x y} + g(y) = x + xe^x y + g(y).$

$\frac{\partial F}{\partial y} = N \Rightarrow xe^x + g'(y) = xe^x + 2 \Rightarrow g(y) = \int 2\, dy = 2y + c.$

$\therefore F = x + xe^x y + 2y + c.$

$\therefore$ An implicit solution is $x + xe^x y + 2y = c.$

$\therefore$ $y(xe^x + 2) = c - x \Rightarrow y = \frac{c - x}{xe^x + 2}$ is an implicit solution.

continued ...

16. [12 marks] Note that there is a **Table of Laplace Transforms** on page 3.

The motion of a spring-mass system that is given a sharp blow at time $t = \pi$ is described by the initial-value problem:

$$y''(t) + 2y'(t) + 10y(t) = K\delta(t - \pi),\ y(0) = 0, y'(0) = 3.$$

For what value of K will the system become motionless, i.e., $y(t) = 0$ for all $t > \pi$.

Want to solve this IVP, & at the end set it equal to zero & solve for K.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = K\mathcal{L}\{\delta(t-\pi)\}$$

$$\Rightarrow [s^2Y(s) - \underbrace{sy(0)}_{0} - \underbrace{y'(0)}_{3}] + 2[sY(s) - \underbrace{y(0)}_{0}] + 10Y(s) = Ke^{-s\pi}$$

$$\Rightarrow s^2Y(s) - 3 + 2sY(s) + 10Y(s) = Ke^{-s\pi}$$

$$\Rightarrow Y(s)[s^2 + 2s + 10] = Ke^{-s\pi} + 3$$

$$\Rightarrow Y(s) = \frac{Ke^{-s\pi} + 3}{(s+1)^2 + 9} \Rightarrow y(t) = K\mathcal{L}^{-1}\left\{\frac{e^{-s\pi}}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$\Rightarrow y(t) = \frac{K}{3}\mathcal{U}(t-\pi)\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}\bigg|_{t-\pi} + e^{-t}\sin(3t)$$

$$= \frac{K}{3}\mathcal{U}(t-\pi)e^{-(t-\pi)}\sin(3(t-\pi)) + e^{-t}\sin(3t)$$

For $t > \pi$

$$= -\frac{K}{3}e^{-t+\pi}\sin(3t) + e^{-t}\sin(3t). \quad y(t) = 0 \ \forall\ t > \pi$$

$$\Rightarrow e^{-t}\sin(3t)\left[-\frac{K}{3}e^{\pi} + 1\right] = 0 \Rightarrow K = 3e^{-\pi}.$$

continued ...

17. [12 marks]

(a) [6 marks] If the *Annihilator Approach* is used to solve the differential equation:

$$y''' - y'' + 4y' - 4y = 5e^x - 2xe^{2x}\sin(3x),$$

then what annihilator should be used? (DO NOT SOLVE.)

(b) [6 marks] Find the general solution of the following differential equation using the *Annihilator Approach*:

$$y'' + y' - 2y = 8 - 40\cos(2x).$$

(a) $(D-1)$ ann. $5e^x$.

$(D^2-4D+13)^2$ ann. $2xe^{2x}\sin(3x)$.

[since $[D^2-2\alpha D+(\alpha^2+\beta^2)]^2$ ann. $c_1 x e^{\alpha x}\sin(\beta x)$ & here $\alpha=2$, $\beta=3$].

$\therefore (D-1)(D^2-4D+13)^2$ ann. $5e^x - 2xe^{2x}\sin(3x)$.

(b) $y''+y'-2y=0$

$m^2+m-2=0$

$(m+2)(m-1)$

$m=-2$, $m=1$.

$y_c = c_1 e^{-2x} + c_2 e^{x}$.

$D(D^2+4)$ ann. $8-40\cos(2x)$.

$D(D^2+4)(D^2+D-2)=0$

$m=0$ $m^2=-4$ $m=1$

$m=\pm 2i$ $m=-2$.

$y_p = C_1 + c_3\cos(2x) + c_4\sin(2x)$.

$y_p' = -2c_3\sin(2x) + 2c_4\cos(2x)$

$y_p'' = -4c_3\cos(2x) - 4c_4\sin(2x)$.

continued ...

$y_p'' + y_p' - 2y_p = 8 - 40\cos(2x)$

Continuation of solution to problem 17.

$\Rightarrow \quad -4c_3\cos(2x) - 4c_4\sin(2x) - 2c_3\sin(2x) + 2c_4\cos(2x)$

$-2c_1 \quad -2c_3\cos(2x) - 2c_4\sin(2x) = 8 - 40\cos(2x)$

$\Rightarrow \quad -2c_1 + \sin(2x)[-4c_4 - 2c_3 - 2c_4]$

$+ \cos(2x)[-4c_3 + 2c_4 - 2c_3] = 8 - 40\cos(2x)$

$\Rightarrow \quad c_1 = -4 \;\;\&\;\; -2c_3 - 6c_4 = 0 \;\;\&\;\; -6c_3 + 2c_4 = -40$

$\Rightarrow c_3 = \frac{6c_4}{-2} = -3c_4 \Rightarrow 18c_4 + 2c_4 = -40 \Rightarrow c_4 = -2 \;\&\; c_3 = 6.$

$\therefore\; y_p = -4 + 6\cos(2x) - 2\sin(2x).$

$\therefore$ The general solution is:

$y = y_c + y_p = c_1e^{-2x} + c_2e^{x} - 4 + 6\cos(2x) - 2\sin(2x).$

continued ...

18. [12 marks] Consider the following differential equation:

$$y'' - y' + 4xy = 0.$$

Find the first 3 nonzero terms for each of two linearly independent solutions in the form of power series about $x_0 = 0$.

$x_0 = 0$ is an ordinary pt, since $P(x) = -1$ & $Q(x) = 4x$, both analytic at $x_0 = 0$.

$y = \sum_{n=0}^{\infty} c_n x^n$.

$$0 = y'' - y' + 4xy = \underbrace{\sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} c_n n x^{n-1}}_{k=n-1} + 4\underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1}$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + 4\sum_{k=1}^{\infty} c_{k-1} x^k$$

$$= 2c_2 - c_1 + \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1) - c_{k+1}(k+1) + 4c_{k-1} \right] x^k$$

$$\Rightarrow c_2 = \frac{c_1}{2}, \quad c_{k+2} = \frac{c_{k+1}(k+1) - 4c_{k-1}}{(k+2)(k+1)}.$$

$$\Rightarrow c_3 = \frac{2c_2 - 4c_0}{3\cdot 2} = \frac{c_1 - 4c_0}{6}, \quad c_4 = \frac{3c_3 - 4c_2}{4\cdot 3} = \frac{\frac{c_1 - 4c_0}{2} - 2c_1}{12}$$

$$= \frac{\frac{1}{2}c_1 - 2c_0 - 2c_1}{12} = \frac{-3}{2\cdot 12}c_1 - \frac{c_0}{6}$$

$$= -\frac{1}{8}c_1 - \frac{1}{6}c_0$$

$c_0 = 1,\ c_1 = 0$:

$$y_1 = 1 - \frac{2}{3}x^3 - \frac{1}{6}x^4 + \dots$$

$c_0 = 0,\ c_1 = 1$:

$$y_2 = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

2 lin. ind. solutions.

[could choose different values for c_0 & c_1].

continued ...

19. [12 marks] Note that there is a **Table of Laplace Transforms** on page 3.

(a) [4 marks] Given that $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a \geq 0$, using the definition of the Laplace transform, prove that if $b > 0$, then $\mathcal{L}\{f(bt)\} = \frac{1}{b}F\left(\frac{s}{b}\right)$, $s > ab$.

(b) [4 marks] Derive an expression for $\mathcal{L}\{\, t \int_0^t f(t-v)g(v)\, dv \,\}$.

(c) [4 marks] Find $\mathcal{L}\{\, t \int_0^t (t-v)^2 \cos(v)\, dv \,\}$. (*Hint:* Use (b)).

a) W.T.S. If $b>0$, then $\mathcal{L}\{f(bt)\} = \frac{1}{b}\mathcal{L}\{\frac{s}{b}\}$, $s>ab$.

Let $u=bt$
$du = b\,dt$
$\frac{1}{b}du = dt$

$\mathcal{L}\{f(bt)\} = \int_0^\infty e^{-st} f(bt)\,dt$

$= \frac{1}{b}\int_0^\infty e^{-s(\frac{1}{b}u)} f(u)\,du$

since $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a \geq 0$

$= \frac{1}{b}\int_0^\infty e^{-(\frac{s}{b})u} f(u)\,du = \frac{1}{b}F(\frac{s}{b})$ for $\frac{s}{b} > a \iff s > ab$.

$\therefore \mathcal{L}\{f(bt)\} = \frac{1}{b}F(\frac{s}{b})$ for $s>ab$. ∎

b) $\mathcal{L}\{t \underbrace{\int_0^t f(t-v)g(v)\,dv}_{f*g}\} = -\frac{d}{ds}\mathcal{L}\{f*g\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$

$= -\frac{d}{ds} F(s)G(s) = -F'(s)G(s) - F(s)G'(s)$.

c) $\mathcal{L}\{t \underbrace{\int_0^t (t-v)^2 \cos(v)\,dv}_{\cos t * t^2}\}$. So, here $F(s) = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$.

$F'(s) = \frac{(s^2+1) - s(2s)}{(s^2+1)^2} = \frac{-s^2+1}{(s^2+1)^2}$.

$G(s) = \mathcal{L}\{t^2\} = \frac{2}{s^3} = 2s^{-3}$.

$G'(s) = -6s^{-4}$. $\therefore \mathcal{L}\{t\int_0^t (t-v)^2 \cos v\,dv\} = -(F'(s)G(s) + F(s)G'(s))$

continued ...

Continuation of solution to problem 19.

$$= -\left[\frac{-s^2+1}{(s^2+1)^2} \cdot \frac{2}{s^3} + \frac{s}{s^2+1} \cdot \frac{-6}{s^4} \right] = \frac{2s^2-2}{s^3(s^2+1)^2} + \frac{6(s^2+1)}{s^3(s^2+1)^2}$$

$$= \frac{8s^2+4}{s^3(s^2+1)^2}.$$

continued ...