MATHEMATICS 2C03 PRACTICE FINAL EXAM

DAY SECTIONS 01, 02
DURATION of FINAL EXAM: 3 HOURS
McMASTER UNIVERSITY

THIS EXAMINATION INCLUDES 20 PAGES AND 20 QUESTIONS. IT IS POSSIBLE TO OBTAIN A TOTAL OF 102 MARKS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR. A **Table of Laplace Transforms** is provided on the top of page 3.

SPECIAL INSTRUCTIONS:
Page 20 is to be used for continuation of a problem if you run out of space. **Ask the invigilators** for **SCRAP paper** if required. **NO NOTES OR AIDS OR PIECES OF PAPER OF ANY KIND (other than that distributed by the invigilator) ARE PERMITTED.**

You are **NOT permitted** to have any **ELECTRONIC DEVICES** of any kind, including calculators and cell phones.

PART I is made up of **14 Multiple choice questions**. **MARK YOUR ANSWERS ON THE OMR EXAMINATION SHEET with HB pencil ONLY.** For this part, only the OMR Examination sheet will be marked. Each multiple choice question is worth **3 marks**. There is **no penalty** for an incorrect answer.

PART II is made up of **5 Complete Answer Questions**. Each question is of equal value.

You must print your name and ID number at the **top of each complete answer page** in the space provided **as well as** on this page below. **You MUST hand in both the OMR EXAMINATION SHEET** and **this examination paper.**

NAME:________________________________

ID #:______________________ Tutorial #:______________________

Questions	Mark	Out of
15		12
16		12
17		12
18		12
19		12
TOTAL		60

GOOD LUCK!

continued ...

OMR EXAMINATION - STUDENT INSTRUCTIONS

NOTE: **IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED: YOUR EXAMINATION RESULT DEPENDS UPON PROPER ATTENTION TO THESE INSTRUCTIONS.**

The scanner, which reads the sheets, senses the shaded areas by their non-reflection of light. A heavy mark must be made, completely filling the circular bubble, with an HB pencil. Marks made with a pen or felt-tip marker will **NOT** be sensed. Erasures must be thorough or the scanner may still sense a mark. Do **NOT** use correction fluid on the sheets. Do **NOT** put any unnecessary marks or writing on the sheet.

1. Print your name, student number, course name, section number and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.

2. Mark your student number in the space provided on the sheet on Side 1 and fill in the corresponding bubbles underneath.

3. Mark only **ONE** choice from the alternatives (1,2,3,4,5 or A,B,C,D,E) provided for each question. If there is a True/False question, enter response of 1 (or A) as True, and 2 (or B) as False. The question number is to the left of the bubbles. Make sure that the number of the question on the scan sheet is the same as the question number on the test paper.

4. Pay particular attention to the Marking Directions on the form.

5. Begin answering questions using the first set of bubbles, marked 1".

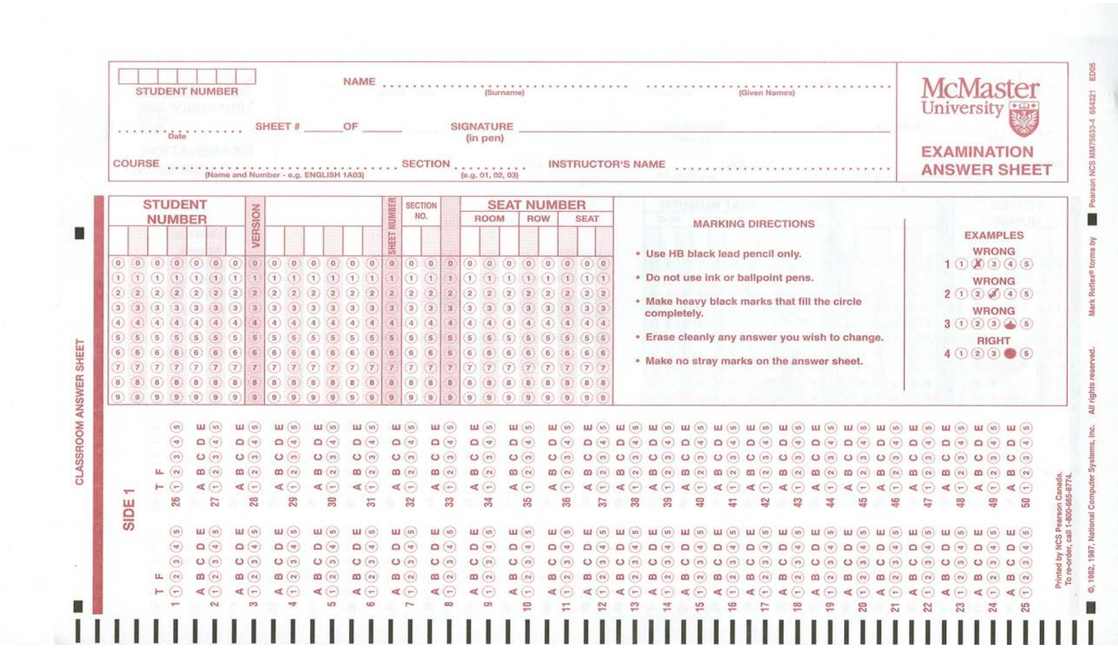

Table of Laplace Transforms

Here $\mathscr{L}\{f(t)\} = F(s)$.

Section 7.1:

- $\mathscr{L}\{1\} = \dfrac{1}{s}$
- $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$
- $\mathscr{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}$
- $\mathscr{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$
- $\mathscr{L}\{\sinh(kt) = \dfrac{k}{s^2-k^2}$
- $\mathscr{L}\{\cosh(kt)\} = \dfrac{s}{s^2-k^2}$

Section 7.2:

- $\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$

Section 7.3:

- $\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}\mid_{s\to s-a} = F(s-a)$, where $a \in \mathbb{R}$
- $\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)\mid_{s\to s-a}\} = e^{at}f(t)$
- $\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$, where $a > 0$
- $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$, where $a > 0$
- $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$, where $a > 0$
- $\mathscr{L}\{\mathscr{U}(t-a)\} = \dfrac{e^{-as}}{s}$, where $a > 0$

continued ...

Section 7.4:

- $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$, $n = 1, 2, \ldots$
- $\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$

Section 7.5:

- $\mathscr{L}\{\delta(t - t_0)\} = e^{-st_0}$, for $t_0 > 0$

PART I: MULTIPLE CHOICE

There are 14 multiple choice questions. All questions have the same value. A correct answer scores 3 and an incorrect answer scores zero. Record your answer on the OMR Examination Answer Sheet provided. Follow the instructions on page 2 carefully. **Use HB pencil only.** Ask the invigilators for scrap paper if required.

1. The differential equation $(2xy + x^2 + x^4)dx + (1 + x^2)dy = 0$, is:

   (A) homogeneous and exact.

   (B) separable and homogeneous.

   (C) separable and exact.

   (D) linear and homogeneous.

   (E) linear and exact.

2. An integrating factor of the form $\mu(x)$ for $(2x^2 + y)dx + x(xy - 1)dy = 0$ is:

   (A) x^2

   (B) x^{-1}

   (C) x^{-2}

   (D) e^x

   (E) xe^x

3. If $y(x)$ is the solution of the initial value problem,
   $x^2y'' - 3xy' + 4y = 0,\ y(1) = 0,\ y'(1) = 2,$ then $y(e)$ equals:

   (A) $2e^2$

   (B) $e^2 + 1$

   (C) $2(e^2 + 1)$

   (D) $e^2 + 2e$

   (E) $e^2 + 2$

continued ...

4. If $y(x)$ is the solution of the initial value problem,
   $y' + \frac{1}{x}y = x^2y^2,\ y(1) = 2$, then $y(2)$ equals:

   (A) $-\frac{3}{2}$
   (B) $-\frac{1}{2}$
   (C) $-\frac{1}{4}$
   (D) 0
   (E) $\frac{2}{3}$

5. Find the general solution of the fourth-order homogeneous linear differential equation

$$y^{(4)} + 18y'' + 81y = 0.$$

   (A) $y(x) = (c_1 + c_2x)\cos(3x) + (c_3 + c_4x)\sin(3x)$.
   (B) $y(x) = c_1\cos(3x) + c_2\sin(3x)$.
   (C) $y(x) = c_1 + c_2x + c_3\cos(3x) + c_4\sin(3x)$.
   (D) $y(x) = c_1e^{3x} + c_2xe^{3x} + c_3\cos(3x) + c_4\sin(3x)$.
   (E) $y(x) = (c_1x + c_2x^2)\cos(3x) + (c_3x + c_4x^2)\sin(3x)$.

continued ...

6. Find the Laplace transform of $(t^2+3)u(t-2)$, where $u(t)$ is the unit step function.

(A) $e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{7}{s}\right)$

(B) $e^{-2s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{5}{s}\right)$

(C) $e^{-2s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{3}{s}\right)$

(D) $e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{9}{s}\right)$

(E) $e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s}\right)$

7. Let $F(s)$ denote the Laplace transform of $f(t)=\int_0^t e^{-\tau}\cos(\tau)\,d\tau$.

Then, $F(2)$ is equal to:

(A) $\frac{1}{10}$

(B) $\frac{3}{20}$

(C) $\frac{1}{2}$

(D) $-\frac{1}{e}$

(E) 0

8. The inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+11}\right\}$ is:

(A) $\cos(\sqrt{2}(t-3))-\frac{3}{2}\sin(\sqrt{2}(t-3))$

(B) $e^{-3t}(\cos(\sqrt{2}\,t)-\frac{3}{\sqrt{2}}\sin(\sqrt{2}\,t))$

(C) $t\cos(\sqrt{2}\,t)\,u(t-\sqrt{2})$

(D) $e^{-3t}(\cos(\sqrt{2}\,t)-\sin(\sqrt{2}\,t))$

(E) $e^{-3t}\cos(\sqrt{2}\,t)$

continued ...

9. Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$, if $F(s) = \frac{s^2}{(s+1)(s^2+4)}$.

(A) $e^{-t} + \sin(2t)$

(B) $\frac{1}{2}(e^{-t} + 3\cos(2t) - 3\sin(2t))$

(C) $\frac{1}{4}(e^{-t} + 2\cos(2t) + 3\sin(2t))$

(D) $\frac{1}{5}(e^{-t} + 4\cos(2t) - 2\sin(2t))$

(E) $e^{-t} + 2\cos(2t) - 2\sin(2t)$

10. The recursive formula for the coefficients a_n of the power series solution about $x_0 = 0$ of the differential equation $y'' - xy = 0$, is given by:

(A) $a_{n+2} = \frac{2na_{n-1}}{(n+2)(n+1)}$, $n \geq 1$.

(B) $a_{n+2} = \frac{2(n-1)a_{n-1}}{n(n+1)}$, $n \geq 1$.

(C) $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$, $n \geq 1$.

(D) $a_{n+2} = \frac{na_{n-1}}{(n+1)}$, $n \geq 1$.

(E) $a_{n+2} = \frac{a_{n-1}}{(n)(n+1)}$, $n \geq 1$.

continued ...

11. Which one of the functions listed below is a power series solution about the ordinary point $x_0 = 0$, of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, with radius of convergence $R = \infty$, for the differential equation $(1+2x^2)y'' + A(x)y' + B(x)y = 0$, given only that $A(x)$ and $B(x)$ are analytic functions, the coefficients a_0 and a_1 are arbitrary, and the recurrence formula is:

$$a_n = \frac{-(2n-5)(n-5)}{n(n-1)}\, a_{n-2}, \quad n \geq 2.$$

(A) $3x + x^3$

(B) $2 + \frac{1}{3}x^3$

(C) $3x + \frac{1}{3}x^3 + \frac{2}{20}x^5$

(D) $1 + \frac{1}{2}x^2$

(E) $\sum_{k=1}^{\infty} (-1)^k \frac{[(-1)(3)(7)\cdots(4k-5)]x^{2k}}{k!\,(2k-3)(2k-1)}$

12. The *best* one can say, without actually solving the differential equation,

$(x^2 + 6x + 8)y'' + 6(x-3)y' + 4y = 0$, is that the radius of convergence of the power series solution about, $x_0 = 3$, is *at least*:

(A) ∞

(B) 6

(C) 5

(D) 4

(E) 3

continued ...

13. A complete list of the *regular singular points* of the differential equation

$$x(x+2)^2(x-5)^2(x-1)^4y'' + (x-5)^2(x+3)(x-1)^3y' + (x-1)^2(x+2)y = 0$$

is given by:

(A) 1, 5

(B) 0, 1

(C) 0, 1, 5

(D) 0, 1, -2, 5

(E) 0, 1, -2, -3, 5

14. The roots of the indicial equation for the point $x_0 = 0$ of the differential equation $x^2y'' + xy' + (x^2-1)y = 0$ are:

(A) -1, 0

(B) -1, $\frac{1}{2}$

(C) $\frac{1}{2}$, $\frac{3}{2}$

(D) -1, 1

(E) 0, $\frac{1}{2}$

continued ...

PART II: COMPLETE ANSWER QUESTIONS

Questions 15-19 Each question is of equal value.

You will be graded on the clarity and presentation of the solution, not just upon whether or not you obtain the correct answer. Scrap paper will NOT be graded. There are sheets at the end of this examnation for continuing solutions. If you need to use them, indicate that your are continuing the problem, and then indicate which problem you are continuing one the pagges at the end of this paper.

15. [12 marks] Find an explicit solution to the following differential equation:

$$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0.$$

continued ...

16. [12 marks] Note that there is a **Table of Laplace Transforms** on page 3.

The motion of a spring-mass system that is given a sharp blow at time $t = \pi$ is described by the initial-value problem:

$$y''(t) + 2y'(t) + 10y(t) = K\delta(t - \pi),\ y(0) = 0, y'(0) = 3.$$

For what value of K will the system become motionless, i.e., $y(t) = 0$ for all $t > \pi$.

continued ...

17. [12 marks]

(a) [6 marks] If the *Annihilator Approach* is used to solve the differential equation:

$$y''' - y'' + 4\,y' - 4\,y = 5e^{x} - 2xe^{2x}\sin(3x),$$

then what annihilator should be used? (DO NOT SOLVE.)

(b) [6 marks] Find the general solution of the following differential equation using the *Annihilator Approach*:

$$y'' + y' - 2\,y = 8 - 40\cos(2x).$$

continued ...

Continuation of solution to problem 17.

18. [12 marks] Consider the following differential equation:

$$y'' - y' + 4xy = 0.$$

Find the first 3 nonzero terms for each of two linearly independent solutions in the form of power series about $x_0 = 0$.

continued ...

Continuation of the solution to problem 18.

continued ...

19. [12 marks] Note that there is a **Table of Laplace Transforms** on page 3.

(a) [4 marks] Given that $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a \geq 0$, using the definition of the Laplace transform, prove that if $b > 0$, then $\mathcal{L}\{f(bt)\} = \frac{1}{b}F\left(\frac{s}{b}\right)$, $s > ab$.

(b) [4 marks] Derive an expression for $\mathcal{L}\{\, t \int_0^t f(t-v)g(v)\; dv \,\}$.

(c) [4 marks] Find $\mathcal{L}\{\, t \int_0^t (t-v)^2 \cos(v)\; dv \,\}$. (*Hint:* Use (b)).

continued ...

Continuation of solution to problem 19.

continued ...

If you need to continue the solution to a problem here, mark clearly on the page where the problem is stated that the "*solution is continued on page 20*", and on this page mark clearly which problem you are continuing. Scrap paper will not be graded.

THE END

Solutions:

1. E
2. C
3. A
4. B
5. A
6. A
7. B
8. B
9. D
10. C
11. A
12. C
13. C
14. D
15. $y = \frac{c-x}{xe^x+2}$
16. $K = 3e^{-\pi}$
17. a) $(D-1)(D^2-4D+13)^2$, b) $y = c_1e^{-2x} + c_2e^x - 4 + 6\cos(2x) - 2\sin(2x)$
18. $y_1 = c_0 - \frac{2}{3}c_0x^3 - \frac{1}{6}c_0x^4 + \ldots$, $y_2 = c_1x + \frac{1}{2}c_1x^2 + \frac{1}{6}c_1x^3 + \ldots$
19. b) $-F'(s)G(s) - F(s)G'(s)$, c) $\frac{8s^2+4}{s^3(s^2+1)^2}$

THE END