MATH 2C03 - Midterm

Duration: 75 Minutes
McMaster University

July 15th, 2015

This test paper includes 9 pages and 6 questions. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of the invigilator.

Special Instructions: Check to see that you have all the pages and that no pages are blank. Page 9 is to be used for scrap paper of the continuation of a problem if you run out of room. Use the back pages for rough work.

You are not permitted to have **ELECTRONIC DEVICES** of any kind, including calculators or cell phones.

No notes or aids or pieces of paper of any kind (other than that distributed by the invigilator) are permitted.

You must print your name and ID number at the **top of each page** in the space provided **as well as** on this page below.

NAME: * marking scheme * ID #:

Questions	Mark	Out of
1		8
2		8
3		9
4		8
5		9
6		8
TOTAL		50

Good Luck!

Fill in the Blanks

1. There are 3 fill in the blank questions. A correct answer scores full points and an incorrect answer scores zero. Record your answer by writing your answer in the blank.

[1pt for each blank]

[**3 marks**] a) Consider the differential equation $y' - ky = A$, where k and A are constants. The critical point $-A/K$ of this equation is a(n) repeller (attractor (stable) or repeller (unstable)) for $k > 0$ and a(n) attractor (attractor (stable) or repeller (unstable)) for $k < 0$.

$y' = A + Ky = 0$
$\Rightarrow y = -A/K.$

$-A/K - 1$	$\boxed{-A/K}$	$-A/K + 1$
$-K$		K

If $K > 0$: $-A/K$ repeller/unstable

If $K < 0$: $-A/K$ stable/attractor

[**2 marks**] b) If the set consisting of two functions, f_1 and f_2, is linearly independent on an interval I, then the Wronskian $W(f_1, f_2) \neq 0$ for all x in I. False [2pts] (True/False)

Theorem 4.1.3 only holds if f_1 & f_2 are solutions to a 2^{nd}-order DE. Therefore, it's possible for this to fail.
e.g. Consider x & $|x|$ on $[-1, 1]$. Can't compute Wronskian at $x = 0$, b/c derivative of $|x|$ not defined there, but these are linearly independent.

[**3 marks**] c) If $y = 3e^x + 6x^2 - x + 1$ is a solution of a linear homogeneous fourth-order differential equation with constant coefficients, then the roots of the auxiliary equation are $m = 0$ (triple root), $m = 1$ (single root). (include what the multiplicity of each root is).

[1pt for $m=0$
1pt for (triple root)
1pt for $m=1$]

Must have auxiliary eq'n $m^3(m-1) = 0$.

[An explanation is not necessary for these fill in the blanks].

Full Answer Questions

In this section, you must show your work to to receive full credit. You will be graded on the clarity and presentation of the solution, not just upon whether or not you obtain the correct solution. The value of each question is given in the margin.

[**4 marks**] 2.a) Find an *explicit* solution of $xy'+y=\sin x$ (be sure to include an interval with your solution).

$$y' + \underbrace{\tfrac{1}{x}}_{P(x)} y = \underbrace{\frac{\sin x}{x}}_{f(x)}$$

[1pt] $\int P(x)dx = \int \frac{1}{x} dx = \ln x.$

[1pt] $y = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\, dx$

$= x^{-1} \int x \frac{\sin x}{x} dx = x^{-1}[-\cos x + c] = -x^{-1}\cos x + c/x.$

$y' = x^{-2}\cos x + x^{-1}\sin x - c/x^2.$ y & y' cont. on any interval not containing zero.

[1pt]

$\therefore y = -\frac{\cos x}{x} + c/x$ on $\underbrace{(0,\infty)}$ is an explicit solution.

[1pt]

any interval not containing zero would be fine

[**4 marks**] 2.b) Determine whether Theorem 1.2.1 (Existence/Uniqueness of 1st-order IVP's) guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the point $(2, -3)$.

[1pt] $F(x,y) = \sqrt{y^2 - 9}$ defined & cont. when $y^2 - 9 \geq 0$

$\Leftrightarrow y^2 \geq 9 \Leftrightarrow y \geq 3$ or $y \leq -3$.

[1pt] $\frac{\partial F}{\partial y} = \frac{1}{2}(y^2-9)^{-\frac{1}{2}} \cdot 2y = \frac{y}{\sqrt{y^2-9}}$. $\frac{\partial F}{\partial y}$ defined & cont.

when $y > 3$ or $y < -3$.

[2pts] In order for the assumptions of Theorem 1.2.1 to hold, we need F & $\frac{\partial F}{\partial y}$ cont. on a region R with $(2,-3)$ in its interior.

$y = -3$ is not in our range for $\frac{\partial F}{\partial y}$ being cont.

$\Rightarrow$ Theorem 1.2.1 does not guarantee a unique solution through $(2,-3)$.

[9 marks] 3. Using Undetermined Coefficients (Annihilator Approach), find a general solution of $y''' - 5y'' + 6y' = 8 + 2\sin x$.

$$m^3 - 5m^2 + 6m = 0$$

$$m[m^2 - 5m + 6] = 0$$

$$m(m-3)(m-2) = 0$$

$$m = 0,\ m = 3,\ m = 2.$$

[3pts] $y_c = \underline{c_1} + \underline{c_2 e^{2x}} + \underline{c_3 e^{3x}}.$

D annihilates 8 & D^2+1 annihilates $2\sin x$.

$$D(D^2+1)[D^3 - 5D^2 + 6D](y) = 0$$

$$m(m^2+1)m(m-3)(m-2)$$

$m = 0$ double

$m = \pm i$

$m = 2$

$m = 3$

$$\cancel{c_1} + c_2 x + \cancel{c_3 e^{2x}} + \cancel{c_4 e^{3x}} + c_5 \sin x + c_6 \cos x$$

$y_p = c_2 x + c_5 \sin x + c_6 \cos x.$ [3pts]

$$y_p' = c_2 + c_5 \cos x - c_6 \sin x$$

$$y_p'' = -c_5 \sin x - c_6 \cos x.$$

$$y_p''' = -c_5 \cos x + c_6 \sin x$$

$y_p''' - 5y_p'' + 6y_p' = 8 + 2\sin x \iff -c_5 \cos x + c_6 \sin x + 5c_5 \sin x + 5c_6 \cos x + 6c_2 + 6c_5 \cos x - 6c_6 \sin x = 8 + 2\sin x$

$\iff (6c_2) + (c_6 + 5c_5 - 6c_6)\sin x + (-c_5 + 5c_6 + 6c_5)\cos x = 8 + 2\sin x$

$\iff 6c_2 = 8$, $c_2 = 4/3$ & $\left.\begin{aligned} 5c_5 - 5c_6 &= 2 \\ 5c_5 + 5c_6 &= 0 \end{aligned}\right\}$ $10c_5 = 2 \Rightarrow c_5 = 1/5 \Rightarrow c_6 = -1/5.$

back of page

$\therefore\ y_p = \frac{4}{3}x + \frac{1}{5}\sin x - \frac{1}{5}\cos x.$

$\therefore$ General solution is

$$y = c_1 + c_2 e^{2x} + c_3 e^{3x} + \frac{4}{3}x + \frac{1}{5}\sin x - \frac{1}{5}\cos x.$$

[3pts]

[8 marks] 4. Find an implicit solution for the initial-value problem
$y + y^4 = (y^3 + 3x)y'$, $y(1) = 1$.

$$\underbrace{(y+y^4)}_{M}dx + \underbrace{(-y^3-3x)}_{N}dy = 0$$

Not exact, since $\frac{\partial M}{\partial y} = 1 + 4y^3 \neq \frac{\partial N}{\partial x} = -3$.

$$\frac{N_x - M_y}{M} = \frac{-3-1-4y^3}{y+y^4} = \frac{-4(y^3+1)}{y(1+y^3)} = -\frac{4}{y}$$ function of y alone.

$$e^{\int -4\frac{1}{y}dy} = e^{-4\ln y} = y^{-4}.$$ [2pts]

$$\underbrace{(y^{-3}+1)}_{\tilde{M}}dx + \underbrace{(-y^{-1}-3xy^{-4})}_{\tilde{N}}dy = 0$$ exact, since $\frac{\partial \tilde{M}}{\partial y} = -3y^{-4} = \frac{\partial \tilde{N}}{\partial x}$.

$$\nabla F = (\tilde{M}, \tilde{N}) \Rightarrow \frac{\partial F}{\partial x} = \tilde{M} \text{ \& } \frac{\partial F}{\partial y} = \tilde{N}.$$

$$\frac{\partial F}{\partial x} = \tilde{M} \Rightarrow F = \int (y^{-3}+1)\,dx = (y^{-3}+1)x + g(y).$$ [2pts]

[2pts]

$$\frac{\partial F}{\partial y} = \tilde{N} \Rightarrow -3y^{-4}x + g'(y) = -y^{-1} - 3xy^{-4} \Rightarrow g = \int -y^{-1}dy = -\ln y + c.$$

$$\therefore (y^{-3}+1)x - \ln y = c.$$

$$y(1) = 1 \Rightarrow 2 = c.$$

$$\therefore (y^{-3}+1)x - \ln y = 2$$ implicit solution. [2pts]

[9 marks] 5. Find the general solution of $x^2y'' + xy' - y = \ln x$.

(Hint: $\int \ln x dx = x\ln x - x + c$).

Cauchy-Euler eq'n. [1pt]

$$m(m-1) + m - 1 = 0$$
$$m^2 - 1 = 0$$
$$m = \pm 1.$$

$$y'' + x^{-1}y' - x^{-2}y = \underbrace{\frac{\ln x}{x^2}}_{F(x)}$$

[3pts] $y_c = c_1 \underbrace{x}_{y_1} + c_2 \underbrace{x^{-1}}_{y_2}$.

Use variation of parameters to find particular solution: [1pt]

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1}.$$

$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ x^{-2}\ln x & -x^{-2} \end{vmatrix} = -x^{-3}\ln x.$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & x^{-2}\ln x \end{vmatrix} = x^{-1}\ln x.$$

$u = \ln x \quad v = -x^{-1}$

$du = \frac{1}{x}dx \quad dv = x^{-2}$

$$u_1 = \int \frac{W_1}{W} dx = \frac{1}{2}\int x^{-2}\ln x\, dx = \frac{1}{2}\left[-x^{-1}\ln x + \int x^{-2} dx\right]$$
$$= -\frac{1}{2}x^{-1}\ln x + \frac{1}{2}\left[-x^{-1}\right] = -\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}.$$

$$u_2 = \int \frac{W_2}{W} = -\frac{1}{2}\int \ln x dx = -\frac{1}{2}x\ln x + \frac{1}{2}x.$$

[3pts]

$$y_p = u_1y_1 + u_2y_2 = -\frac{1}{2}\ln x - \frac{1}{2} - \frac{1}{2}\ln x + \frac{1}{2} = -\ln x.$$

$\therefore$ General solution is $y = c_1x + c_2x^{-1} - \ln x$. [1pt]

[8 marks] 6. Solve $x^2y' = xy + y^2 + x^2$. Put your answer in the form $y(x) = \dots$, but don't worry about including an interval where the solution is defined.

$$\underbrace{(xy+y^2+x^2)}_{M}dx - \underbrace{x^2}_{N}dy = 0.$$

[1pt] Homogeneous of degree 2:

$$M(tx,ty) = t^2M(x,y) \text{ \& } N(tx,ty) = t^2N(x,y).$$

$y = ux$

$dy = xdu + udx.$ [2pts]

$$(ux^2+u^2x^2+x^2)dx - x^3du - x^2udx = 0$$

$$x^2(u^2+1)dx - x^3du = 0$$

[3pts] $\displaystyle\int \frac{1}{x}dx = \int\frac{1}{u^2+1}du$

$u = \tan w \Rightarrow w = \tan^{-1}u.$

$du = \sec^2 w\,dw$

$\ln x = \int dw$

$\ln x = w + c$

$\ln x = \tan^{-1}u + c$

$\ln x = \tan^{-1}(y/x) + c$

$y = x\tan(\ln x - c).$ [2pts]