Math 2C03 - Class #9

7.6: Systems of Linear DE's

We can solve a system of linear DE's by taking $\mathcal{L}$ of each eq'n, solve the system of algebraic eq'n's, & then apply $\mathcal{L}^{-1}$.

e.g.) $\begin{cases} x'' + x' + y' = 0 \\ y'' + y' - 4x' = 0 \end{cases}$ $x(0)=1,\ x'(0)=0,\ y(0)=-1,\ y'(0)=5.$

$\Rightarrow \begin{cases} \mathcal{L}\{x''\} + \mathcal{L}\{x'\} + \mathcal{L}\{y'\} = 0 \\ \mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 4\mathcal{L}\{x'\} = 0 \end{cases}$

$$\Rightarrow \begin{cases} [s^2 X(s) - s\underbrace{X(0)}_{1} - \underbrace{X'(0)}_{0}] + [sX(s) - \underbrace{X(0)}_{1}] + [sY(s) - \underbrace{y(0)}_{-1}] = 0 \\ [s^2 Y(s) - s\underbrace{y(0)}_{-1} - \underbrace{y'(0)}_{5}] + [sY(s) - \underbrace{y(0)}_{-1}] - 4[sX(s) - \underbrace{X(0)}_{1}] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} [s^2+s]X(s) + sY(s) = s + \cancel{1} - \cancel{1} \\ -4s\,X(s) + [s^2+s]Y(s) = -s + 5 - \cancel{1} - \cancel{4} \end{cases}$$

$$\Rightarrow \begin{cases} (s+1)X(s) + Y(s) = 1 \\ -4X(s) + (s+1)Y(s) = -1 \end{cases} \Rightarrow \begin{bmatrix} s+1 & 1 \\ -4 & s+1 \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{1}{(s+1)^2+4}\begin{bmatrix} s+1 & -1 \\ 4 & s+1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{(s+1)^2+4}\begin{bmatrix} s+2 \\ -s+3 \end{bmatrix}$$

$$\Rightarrow X(s) = \frac{s+2}{(s+1)^2+4} \quad \& \quad Y(s) = \frac{-s+3}{(s+1)^2+4}$$

$$\Rightarrow X(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\Big|_{s\to s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\Big|_{s\to s+1}\right\}$$

$$= e^{-t}\cos(2t) + \tfrac{1}{2}e^{-t}\sin(2t).$$

$$y(t) = -\mathcal{L}^{-1}\left\{\frac{s-3}{(s+1)^2+4}\right\} = -\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2+4}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\Big|_{s\to s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\Big|_{s\to s+1}\right\}$$

$$= -e^{-t}\cos(2t) + 2e^{-t}\sin(2t).$$

Midterm:

- The midterms will be graded out of 45 (instead of 50).

Take your mark & multiply by $\frac{100}{45}$.

Average: 61 % [insead of 55 %]

Median: 58 % [instead of 52%].

Percent	Total (after scaling)	Total (before scaling)
90 - 100	5	0
80 - 89	6	6
70 - 79	4	5
60 - 69	5	7
50 - 59	10	6
40 - 49	7	11
30 - 39	4	6
20 - 29	1	0
10 - 19	1	2
Absent	5	5

Ch. 6: Series Solutions of Linear Eq$^{\text{n}}$'s

So far in this course, we have primarily looked at 1st-order DE's (Ch. 2) & higher-order linear DE's with constant coefficients (Ch. 4 & Ch. 7). We did look at Cauchy-Euler eq$^{\text{n}}$'s which had the form $a_n x^n y^{(n)} + \dots + a_1 x y' + a_0 y = g(x)$ (Ch. 4.7).

In this Chapter, we'll look at 2nd-order linear eq$^{\text{n}}$'s whose coefficients are polynomials in x. For many DE's of this form, it turns out that they don't possess solutions which are elementary functions. However, many do possess solutions defined by infinite series.

6.1: Review of Power Series

Def$^{\text{n}}$: A power series centered at x_0 has the form:

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots .$$

Def$^{\text{n}}$: A power series converges if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-x_0)^n$$ exists, i.e., if the sequence of partial sums $\{S_N(x)\}$ converges.

If the limit does not exist at x, then the series is divergent.

Def$^{\text{n}}$: A power series converges absolutely at x if

$$\sum_{n=0}^{\infty} |c_n| |x-x_0|^n$$ converges.

[absolutely convergent $\Rightarrow$ convergent].

$\boxed{\text{Theorem}}$ [<u>Radius of Convergence</u>]: Given a power series $\sum_{n=0}^{\infty} c_n (x-x_0)^n$ centered at x_0, there are only 3 possibilities:

$\boxed{i}$ The series converges only when $x = x_0$. $[R=0]$

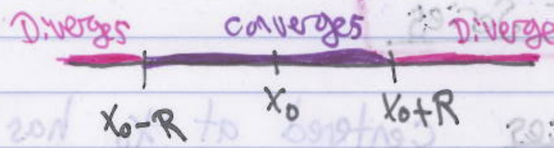
$\boxed{ii}$ The series converges for all x. $[R=\infty]$

$\boxed{iii}$ There's a positive number R s.t. the series converges if $|x-a| < R$ & diverges if $|x-a| > R$.

$\boxed{\text{Def}^{n}}$: The number R is called the <u>radius of convergence</u>. The <u>interval of convergence</u> is the interval that consists of all values of x for which the series converges.

have to check endpoints

$\boxed{\text{e.g.}}$ In $\boxed{i}$ $[x_0, x_0]$, in $\boxed{ii}$ $(-\infty, \infty)$, in $\boxed{iii}$ (x_0-R, x_0+R) or $(x_0-R, x_0+R]$, or $[x_0-R, x_0+R)$, or $[x_0-R, x_0+R]$.



i.e. If x is in the interval of convergence & isn't an endpoint, then the power series converges absolutely at x.

$\boxed{\text{Ratio Test:}}$ Consider the power series $\sum_{n=0}^{\infty} c_n (x-x_0)^n$. Suppose $\exists$ an integer $N \geq 0$ s.t. $c_n \neq 0 \ \forall n \geq N$.

If $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$, then $R = \frac{1}{L}$. If $L = 0 \Rightarrow R = \infty$

If $L = \infty \Rightarrow R = 0$.

* This test is inconclusive at end points.*

$\boxed{\text{Alternating Series Test:}}$ If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, $b_n > 0$ satisfies:

$\boxed{i}$ $b_{n+1} \leq b_n \ \forall n$

$\boxed{ii}$ $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

$\boxed{\text{e.g. 7}}$ Find the interval & radius of convergence for $\boxed{a}$ $\sum_{n=1}^{\infty} n!\,x^n$.

Use the Ratio test. Here $c_{n+1} = (n+1)!$ & $c_n = n!$.

So, $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(n+1)!}{n!}\right| = |n+1| = \infty$. $\therefore R = 0$ & the interval of convergence is the point $[0,0]$.

$\boxed{b}$ $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$.

Here we have $\lim_{n\to\infty} \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \lim_{n\to\infty} \left|\frac{n}{n(1/n+1)}\right| = 1$.

$\therefore R = 1$. We know it's convergent on $(2,4)$. Now we ~~need to~~ check the endpoints 2 & 4:

$\boxed{\text{Recall:}}$ The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. [b/c $S_{2^n} = 1 + \frac{n}{2}$, so $S_{2^n} \to \infty$ as $n \to \infty$].

$x=2$: If we put $x=2$ into the power series it becomes: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$. Let's try the Alternating Series Test: $b_n = \frac{1}{n} > 0$.
$b_{n+1} = \frac{1}{n+1}$. $n+1 > n \Rightarrow \frac{1}{n} > \frac{1}{n+1} \Rightarrow b_n \geq b_{n+1}\ \forall n$. ✓

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$. $\therefore$ convergent at $x=2$.

$x=4$: $\sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series, & hence diverges.

$\therefore$ The interval of convergence is $[2,4)$.

• A power series defines a function:

$F(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$, whose domain is the interval of convergence.

If $R > 0$, F is cont., diff., & integrable on $(x_0 - R, x_0 + R)$.
If $R = \infty$, F is cont., diff., & integrable on $(-\infty, \infty)$.

$F'(x)$ found by term-by-term differentiation.
$\int F(x)\,dx$ found by term-by-term integration.

$\boxed{\text{e.g.}}$ Suppose $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$.
Find y' & y''.

We'll use this in 6.2

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} c_n n x^{n-1}.$$

$$y'' = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}.$$

• $\boxed{\text{Identity Property}}$: If $\sum_{n=0}^{\infty} c_n (x-x_0)^n = 0$, $R > 0$, $\forall x$ in some open interval, then $c_n = 0$ $\forall n$.

$\boxed{\text{Def}^{\text{n}}}$: A function F is said to be analytic at a point x_0 if it can be represented by a power series in $x - x_0$ with a positive or infinite radius of convergence. [i.e. F analytic if it's infinitely diff. & Taylor series at any pt x_0 in the domain converges to $F(x)$ for x in a nbhd of x_0].

$\boxed{\text{e.g.}}$ e^x, $\sin x$, & $\cos x$ can be represented by Taylor series

$$\sum_{n=0}^{\infty} \frac{F^{(n)}(x_0)}{n!} (x-x_0)^n = F(x_0) + F'(x_0)(x-x_0) + \frac{F''(x_0)}{2!}(x-x_0)^2 + \cdots.$$

$\phi(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$

is smooth but not analytic on $\mathbb{R}$.

$\boxed{\text{e.g.}}$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. $R = \infty$.

[by the Ratio Test, $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0$.]

$\boxed{\text{e.g.}}$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$. $R = \infty$.

$\boxed{\text{e.g.}}$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. $R = \infty$.

$\boxed{\text{e.g.}}$ All polynomials are analytic. [In particular, at $x=0$, any polynomial is equal to its MacLaurin series ($x_0 = 0$)]. $R = \infty$.

$\boxed{\text{e.g.}}$ Constant functions $f(x) = c$ are analytic. $R = \infty$.
[$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + 0 + \dots + 0 + \dots = c = f(x)$].

f :: continuous

$\boxed{\text{Notice:}}$ By def$^{\text{in}}$, analytic functions are infinitely differentiable. So, if a function fails to be continuous at a point $x_0 \Rightarrow$ not analytic at that point.

$\boxed{\text{e.g.}}$ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$. $R = 1$.

* These are all Maclaurin series [$x_0 = 0$]. In Ch. 6, we'll primarly be interested in asking whether or not a function is analytic at $x = 0$, so these Maclaurin series will suffice. *

$\boxed{\text{e.g.}}$ A rational function $\frac{a_1(x)}{a_2(x)}$, where $a_1(x)$ & $a_2(x)$ are polynomials with no common factors, is analytic except at the points where $a_2(x) = 0$.

[e.g.] $\cdot \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$. $[-1,1]$

$\cdot \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$. $(-\infty,\infty)$

$\cdot \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$. $(-\infty,\infty)$

$\cdot \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$, $(-1,1]$.

• **Arithmetic of Power Series:** You can add, multiply, & divided power series in a manner similar to how we add, multiply, & perform long division with polynomials.

[e.g.] Find the Maclaurin series of xe^{3x}.

$$xe^{3x} = [x]\left[\sum_{n=0}^{\infty} \frac{1}{n!}(3x)^n\right] = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{n+1}.$$

[e.g.] Write $\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^{n+2}$ as a single power series.

Let $k = n-1$ in the first summand & $k = n+2$ in the second. i.e. $n = k+1$ in the first & $n = k-2$ in the second. Then

$$\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k + 3\sum_{k=2}^{\infty} c_{k-2} x^k$$

$$= c_1 + 2c_2 x + \sum_{k=2}^{\infty} [(k+1)c_{k+1} + 3c_{k-2}] x^k.$$

$\boxed{\text{e.g.7}}$ Find a power series solution about $x=0$ to $y'+2xy=0$.

[Of course, it would be easier to solve by separating variables or using the 1st-order linear DE formula: $y=e^{-\int 2x\,dx}\left[\int e^{\int 2x\,dx}(0)\,dx\right]=ce^{-x^2}$. But let's do this example to see how it generalizes to the 2nd-order case].

__Guess__: $y=\sum_{n=0}^{\infty}a_n x^n$. Want to determine the coefficients a_n!

$$y'=\sum_{n=1}^{\infty}n a_n x^{n-1}.$$

__Sub. y & y' into DE__: $y'+2xy=\sum_{n=1}^{\infty}n a_n x^{n-1}+2x\sum_{n=0}^{\infty}a_n x^n$

$$=\underbrace{\sum_{n=1}^{\infty}n a_n x^{n-1}}_{k=n-1}+\underbrace{\sum_{n=0}^{\infty}2a_n x^{n+1}}_{k=n+1}$$

$$=\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k+\sum_{k=1}^{\infty}2a_{k-1}x^k$$

$$=a_1+\sum_{k=1}^{\infty}\left[(k+1)a_{k+1}+2a_{k-1}\right]x^k=0$$

by the Identity property

$$\Rightarrow a_1=0 \quad \& \quad \underbrace{(k+1)a_{k+1}+2a_{k-1}=0}$$

$$\Rightarrow a_{k+1}=\frac{-2a_{k-1}}{k+1}$$

$\Rightarrow$ __k=1__: $a_2=\frac{-2a_0}{2}=-a_0$

__k=2__: $a_3=\frac{-2a_1}{4}=0$ ($a_1=0$)

__k=3__: $a_4=\frac{-2a_2}{4}=\frac{1}{2}a_0$ ($a_2=-a_0$)

__k=4__: $a_5=\frac{-2a_3}{5}=0$

__k=5__: $a_6=\frac{-2a_4}{6}=-\frac{1}{6}a_0$

__k=6__: $a_7=\frac{-2a_5}{7}=0.$

__k=7__: $a_8=\frac{-2a_6}{8}=\frac{1}{24}a_0$

We can see that $a_{2n+1} = 0$ & $a_{2n} = \frac{(-1)^n}{n!} a_0$.

Our guess what $y = \sum_{n=0}^{\infty} a_n x^n$, so subbing this in we get:

$$y = a_0 - a_0 x^2 + \frac{1}{2} a_0 x^4 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a_0 X^{2n}.$$

Looking at our table of Maclaurin series, we can see

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} X^{2n} \Rightarrow y = a_0 e^{-x^2},$$ as we expected.

We could apply the Ratio Test here to find the interval of convergence:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{1}{(n+1)!}}{(-1)^n \frac{1}{n!}} \right| = \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0 \Rightarrow R = \infty \Rightarrow$$ interval of convergence is $(-\infty, \infty)$.