Math 2C03 - Class #8

Last Time: We looked at 2 operational properties of $\mathcal{L}$:

$\boxed{1}$ $\mathcal{L}\{e^{at}F(t)\} = F(s-a) = \mathcal{L}\{F(t)\}|_{s\to s-a}$.

$\boxed{2}$ $\mathcal{L}\{F(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$, where $\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$ $a > 0$.

From this we computed:

$\boxed{3}$ $\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$. $\boxed{4}$ $\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$.

Let's do a few more examples using these properties:

Using $\boxed{1}$: **e.g.** Find $\mathcal{L}\{e^{2t}\sin(3t)\}$.

Let $a=2$, $F(t) = \sin(3t)$. Use $\boxed{1}$.

$$\mathcal{L}\{e^{2t}\sin(3t)\} = \mathcal{L}\{\sin(3t)\}|_{s\to s-2} = \frac{3}{s^2+9}\Big|_{s\to s-2} = \frac{3}{(s-2)^2+9}.$$

e.g. Find $\mathcal{L}^{-1}\left\{\frac{6}{(s-5)^4}\right\}$.

The inverse form of $\boxed{1}$ is: $\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}F(t)$.

$$\mathcal{L}^{-1}\left\{\frac{6}{(s-5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{s^4}\Big|_{s\to s-5}\right\} = e^{5t}t^3, \text{ since } \mathcal{L}\{t^3\} = \frac{3!}{s^4}.$$

e.g. Solve $2y'' + 20y' + 51y = 0$, $y(0) = 2$, $y'(0) = 0$.

$$2[s^2Y(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{0}] + 20[sY(s) - \underbrace{y(0)}_{2}] + 51Y(s) = 0$$

$$\Rightarrow Y(s)[2s^2 + 20s + 51] - 4s - 40 = 0$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{2s+20}{s^2+10s+25.5}\right\}$$ We can complete this by completing the square in the denominator & using 1st-Translation Property [1]:

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{2(s+10)}{(s+5)^2+\frac{1}{2}}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+5)+10}{(s+5)^2+\frac{1}{2}}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s+5}{(s+5)^2+\frac{1}{2}}\right\} + 10\,\mathcal{L}^{-1}\left\{\frac{1}{(s+5)^2+\frac{1}{2}}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{1}{2}}\Big|_{s\to s+5}\right\} + 10\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{1}{2}}\Big|_{s\to s+5}\right\}$$

$$= 2e^{-5t}\cos\left(\frac{t}{\sqrt{2}}\right) + 10\,e^{-5t}\sqrt{2}\,\mathcal{L}^{-1}\left\{\frac{\frac{1}{\sqrt{2}}}{s^2+(\frac{1}{\sqrt{2}})^2}\right\}$$

$$= 2e^{-5t}\cos\left(\frac{t}{\sqrt{2}}\right) + 10\sqrt{2}\,e^{-5t}\sin\left(\frac{t}{\sqrt{2}}\right).$$

Using [2] & [3]:

e.g. 7 Find $\mathcal{L}\{e^{2-t}\,\mathcal{U}(t-2)\}$.

Use [2] with $a=2$ & $F(t-a) = e^{2-t} = e^{-(t-2)} \Rightarrow F(t) = e^{-t}$.

$\mathcal{L}\{F(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s),\ a>0.\quad F(s) = \mathcal{L}\{F(t)\}.$

$$\mathcal{L}\{e^{2-t}\,\mathcal{U}(t-2)\} = e^{-2s}\,\underbrace{\mathcal{L}\{e^{-t}\}}_{F(s)} = e^{-2s}\,\frac{1}{s+1} = \frac{e^{-2s}}{s+1}.$$

$$\overset{\shortparallel}{\mathcal{L}\{e^{-(t-2)}\,\mathcal{U}(t-2)\}}$$

e.g. 7 Find $\mathcal{L}\{(3t+1)\,\mathcal{U}(t-1)\}$.

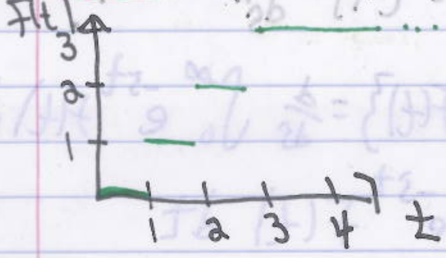
$$\mathcal{L}\{(3t+1)\mathcal{U}(t-1)\} = \mathcal{L}\{[3(t-1)+4]\,\mathcal{U}(t-1)\}$$

$F(t-1) = t-1 \Rightarrow F(t) = t.$

$$= 3\,\mathcal{L}\{(t-1)\,\mathcal{U}(t-1)\} + 4\,\mathcal{L}\{\mathcal{U}(t-1)\} = 3e^{-s}\,\mathcal{L}\{t\} + 4\frac{e^{-s}}{s}$$

$$= \frac{3e^{-s}}{s^2} + \frac{4e^{-s}}{s}.$$

Using $\boxed{3}$: $\boxed{\text{e.g.7}}$ Find the Laplace transform of $f(t)$, where:



Here $f(t) = \mathcal{U}(t-1) + \mathcal{U}(t-2) + \mathcal{U}(t-3)$.

$$\mathcal{L}\{f(t)\} = \frac{e^{-s} + e^{-2s} + e^{-3s}}{s}.$$

Using $\boxed{4}$: $\boxed{\text{e.g.7}}$ Find $\mathcal{L}\{t^2\,\mathcal{U}(t-4)\}$.

Use $\boxed{4}$: $\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}$, $a>0$.

Here $a=4$, $g(t)=t^2$.

$$\mathcal{L}\{t^2\,\mathcal{U}(t-4)\} = e^{-4s}\,\mathcal{L}\{(t+4)^2\} = e^{-4s}\,\mathcal{L}\{t^2+8t+16\}$$

$$= e^{-4s}\left[\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right].$$

7.4: Operational Properties II

Theorem 7.4.1: [Derivatives of Transforms] If $F(s) = \mathcal{L}\{f(t)\}$ & $n = 1, 2, 3, \ldots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Idea Behind Proof: $\frac{d}{ds} F(s) = \frac{d}{ds} \mathcal{L}\{f(t)\} = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt$

$= \int_0^\infty \frac{\partial}{\partial s} [e^{-st} f(t)]\, dt = -\int_0^\infty t\, e^{-st} f(t)\, dt$

$= -\mathcal{L}\{t f(t)\}.$

i.e., $\mathcal{L}\{t f(t)\} = -\frac{d}{ds} \mathcal{L}\{f(t)\}.$

Then, $\mathcal{L}\{t^2 f(t)\} = \mathcal{L}\{t \cdot (t f(t))\} = -\frac{d}{ds}\Big(\underbrace{-\frac{d}{ds}\mathcal{L}\{f(t)\}}_{\mathcal{L}\{t f(t)\}}\Big)$

take $f(t) = t f(t)$

$= \frac{d^2}{ds^2} \mathcal{L}\{f(t)\}.$

⋮ etc.

e.g., $\mathcal{L}\{t^3 e^t\} = (-1)^3 \frac{d^3}{ds^3} \mathcal{L}\{e^t\} = -\frac{d^3}{ds^3} \frac{1}{(s-1)}$

$= -\frac{d^2}{ds^2}[-(s-1)^{-2}] = \frac{d}{ds}[-2(s-1)^{-3}] = \frac{6}{(s-1)^4}.$

[Notice, could have also solved this using 1^{st}-translation Property:
$\mathcal{L}\{t^3 e^t\} = \mathcal{L}\{t^3 \mid s \to s-1\} = \frac{3!}{s^4}\Big|_{s\to s-1} = \frac{6}{(s-1)^4}$].

Exercise: Show $\mathcal{L}\{t^2 \cos t\} = \dfrac{2s^3 - 6s}{(s^2+1)^3}.$

Transforms of Integrals:

Def'n: If F & g are piecewise continuous on $[0,\infty)$, then the convolution of $F(t)$ & $g(t)$ is defined to be:

$$\underbrace{F * g}_{\text{a star...}} := \int_0^t F(\tau)\, g(t-\tau)\, d\tau.$$

not multiplication

i.e., $F*g$ is a function of t.

e.g., Let $F(t) = \cos t$, $g(t) = \sin t$. Then,

$$F*g = \int_0^t \cos(\tau)\, \underbrace{\sin(t-\tau)}_{\sin t\cos\tau - \sin\tau\cos t}\, d\tau$$

$$= \int_0^t \sin t\, \underbrace{\cos^2\tau}_{\frac{1}{2}(1+\cos(2\tau))} - \cos t\, \underbrace{\sin\tau\cos\tau}_{u=\sin\tau,\ du=\cos\tau\, d\tau}\, d\tau$$

$$= \frac{1}{2}\sin t \int_0^t 1 + \cos(2\tau)\, d\tau - \cos t \int u\, du$$

$$= \frac{1}{2}\sin t\left[\tau + \frac{1}{2}\sin(2\tau)\right]\Big|_0^t - \frac{1}{2}\cos t\, \sin^2\tau\Big|_0^t$$

$$= \frac{1}{2}\sin t\left[t + \frac{1}{2}\underbrace{\sin(2t)}_{2\sin t\cos t}\right] - \frac{1}{2}\cos t\, \sin^2 t$$

$$= \frac{1}{2}t\sin t + \frac{1}{2}\sin^2 t\, \cos t - \frac{1}{2}\cos t\, \sin^2 t$$

$$= \frac{1}{2}t\sin t.$$

Theorem 7.4.2 [Convolution Theorem]: If F & g are piecewise cont. on $[0,\infty)$ & of exponential order, then $\mathcal{L}\{F*g\} = \mathcal{L}\{F(t)\}\,\mathcal{L}\{g(t)\} = F(s)G(s)$.

Proof: [see pg. 303]

i.e.7 $\mathcal{L}^{-1}\{F(s)G(s)\} = f*g.$

Theorem: $f*g = g*f.$

Proof: Using change of variables $u = t-\tau$ we have:

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_t^0 f(t-u)\, g(u)\, (-du)$$

$u = t-\tau$ when $\tau = 0 \Rightarrow u = t$

$du = -d\tau$ when $\tau = t \Rightarrow u = 0.$

$$= \int_0^t f(t-u)\, g(u)\, du$$

$$= g*f.$$

$\boxed{\text{e.g.7}}$ Solve $y'' + y = 2\cos t$, $y(0) = 0$, $y'(0)$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 2\mathcal{L}\{\cos t\}$$

$$\Rightarrow s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + Y(s) = \frac{2s}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{2s}{(s^2+1)^2}$$

$$\Rightarrow y(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = 2\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s^2+1)}}_{\mathcal{L}\{\sin t\}} \cdot \underbrace{\frac{s}{(s^2+1)}}_{\mathcal{L}\{\cos t\}}\right\}$$

$$= 2\mathcal{L}^{-1}\{\mathcal{L}\{\sin t\}\mathcal{L}\{\cos t\}\}$$

$$= 2\mathcal{L}^{-1}\{\mathcal{L}\{\sin t * \cos t\}\} = 2\underbrace{(\sin t * \cos t)}_{\frac{1}{2}t\sin t} = t\sin t.$$

$\boxed{\text{e.g.}}$: Evaluate $\mathcal{L}\left\{\int_0^t \tau \sin\tau \, d\tau\right\}$.

Let $F(t) = t\sin t$, $g(t) = 1$. Then, $F*g = \int_0^t \tau\sin\tau \, d\tau$.

So, $\mathcal{L}\left\{\int_0^t \tau\sin\tau\, d\tau\right\} = \mathcal{L}\{t\sin t * 1\} = \mathcal{L}\{t\sin t\}\,\mathcal{L}\{1\}$

$= \left[-\frac{d}{ds}\mathcal{L}\{\sin t\}\right]\left[\frac{1}{s}\right] = \left[-\frac{d}{ds}\frac{1}{s^2+1}\right]\left[\frac{1}{s}\right] = \frac{2s}{(s^2+1)^2}\frac{1}{s} = \frac{2}{(s^2+1)^2}$.

$\boxed{\text{e.g.}}$: Solve for $F(t)$, where $F(t) = \underbrace{\cos t + \int_0^t e^{-\tau} F(t-\tau)\, d\tau}$.

Let $g(t) = e^{-t}$. Then
$g*F = \int_0^t e^{-\tau} F(t-\tau)\, d\tau$.

An eq'n of this type is called a Volterra integral eq'n. i.e.: $F(t) = g(t) + \int_0^t f(\tau) h(t-\tau)\, d\tau$

So, $F(t) = \cos t + e^{-t} * F(t)$

$\Rightarrow \mathcal{L}\{F(t)\} = \mathcal{L}\{\cos t\} + \mathcal{L}\{e^{-t} * F(t)\}$

$\Rightarrow F(s) = \frac{s}{s^2+1} + \underbrace{\mathcal{L}\{e^{-t}\}}_{\frac{1}{s+1}}\underbrace{\mathcal{L}\{F(t)\}}_{F(s)}$

$\Rightarrow F(s)\left[1 - \frac{1}{s+1}\right] = \frac{s}{s^2+1} \Rightarrow F(s)\left[\frac{s}{(s+1)}\right] = \frac{s}{s^2+1}$

$\Rightarrow F(s) = \frac{s+1}{s^2+1} \Rightarrow F(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$\Rightarrow F(t) = \cos t + \sin t$.

$\boxed{\text{e.g.:}}$ Solve $\underbrace{y' + 6y(t) + 9\int_0^t y(\tau)\,d\tau = 1}$, $y(0) = 0$.

An eq^{n} of this type is called an <u>integrodifferential eq^{n}</u>. i.e.: $L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t)$.

(LRC)

* used in Series circuits *

$$\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} + 9\mathcal{L}\{y * 1\} = \mathcal{L}\{1\}$$

$$\Rightarrow s Y(s) - \underbrace{y(0)}_{0} + 6Y(s) + 9\mathcal{L}\{y\}\mathcal{L}\{1\} = \frac{1}{s}$$

$$\Rightarrow Y(s)[s+6] + \frac{9}{s}Y(s) = \frac{1}{s}$$

$$\Rightarrow Y(s)\left[\frac{s^2+6s+9}{s}\right] = \frac{1}{s}$$

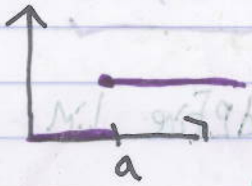
$$\Rightarrow Y(s) = \frac{1}{(s+3)^2} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s+3}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

1st Translation Property

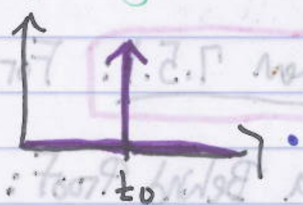
$$\Rightarrow y(t) = t\,e^{-3t}.$$

7.5: The Dirac Delta Function

In 7.3, we discussed how many physical systems behave like the unit step functions $\mathcal{U}(t-a)$:



Similarly, many physical systems (e.g. strike of lightning, wack of golf club) behave like:



This type of behavior can be described by a "generalized function" called the Dirac Delta Function.

Def^{n}: The Dirac Delta function, δ, is characterized by 2 properties:

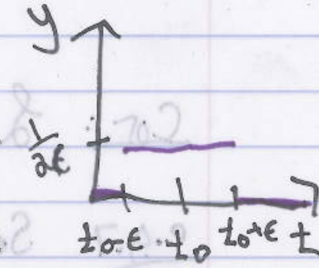
$$\boxed{1}\ \delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases}$$

&

$$\boxed{2}\ \int_0^\infty \delta(t-t_0)\,dt = 1.$$

Intuition: Consider the function

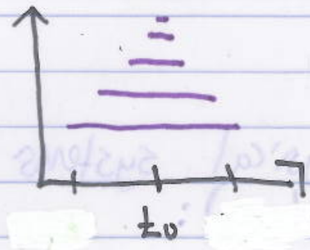
$$\delta_\epsilon(t-t_0)=\begin{cases}\frac{1}{2\epsilon} & \text{if } t_0-\epsilon\le t< t_0+\epsilon.\\ 0, & \text{otherwise.}\end{cases}$$



unit impulse function

Notice that $\int_0^\infty \delta_\epsilon(t-t_0)\,dt = \int_{t_0-\epsilon}^{t_0+\epsilon}\frac{1}{2\epsilon}\,dt = \frac{1}{2\epsilon}[2\epsilon]=1.$

Consider the behavior of $\delta_\epsilon(t-t_0)$ as $\epsilon\to 0$:



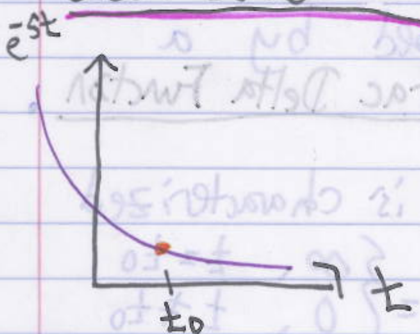
We can see $\lim_{\epsilon \to 0} \delta_\epsilon(t-t_0) = \delta(t-t_0)$.

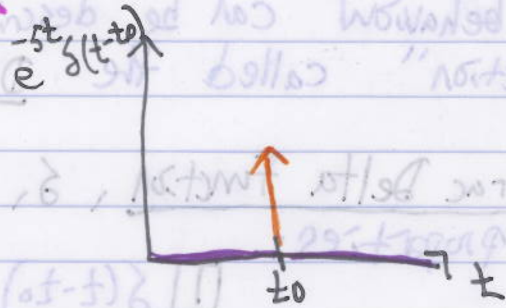
And $\lim_{\epsilon\to 0} \underbrace{\int_0^\infty \delta_\epsilon(t-t_0)\,dt}_{1} = \lim_{\epsilon \to 0}(1) = 1$.

Motivates why we define $\int_0^\infty \delta(t-t_0)\,dt = 1$.

$\boxed{\text{Theorem 7.5.1:}}$ For $t_0 > 0$, $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$.

Idea Behind Proof: $\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt$





zero everywhere except at $t = t_0$.

$e^{-t_0 t} \cdot \infty$

$= e^{-st_0} \underbrace{\int_0^\infty \delta(t-t_0)\,dt}_{=1} = e^{-st_0}$.

[More formal proof on pg. 313].

Cor: $\mathcal{L}\{\delta(t)\} = 1$.

e.g. 7 Solve $y'' + 16y = \delta(t-2\pi)$, $y(0)=0$, $y'(0)=0$.

$\mathcal{L}\{y''\} + 16\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2\pi)\}$

$\Rightarrow s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 16Y(s) = e^{-s2\pi}$

$\Rightarrow Y(s) = e^{-2\pi s}\frac{1}{s^2+16} \Rightarrow y(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{4}{s^2+4^2}\right\} = \frac{1}{4}\sin(4(t-2\pi))\mathcal{U}(t-2\pi)$

2nd Translation Property, $a = 2\pi$.