

$$\begin{aligned} \text{4 } y(t) &= \frac{1}{10} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - \frac{1}{10} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} - \frac{1}{30} \mathcal{L}^{-1} \left\{ \frac{3}{s^2+9} \right\} \\ &= \frac{1}{10} e^{-t} - \frac{1}{10} \cos(3t) - \frac{1}{30} \sin(3t). \end{aligned}$$

## Math 2C03 - Class # 7

### 7.3: Operational Properties I

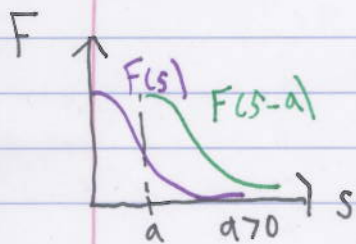
In this section we'll explore several properties of  $\mathcal{L}$  to build a more extensive list of  $\mathcal{L}$  transforms.

**Theorem 7.3.1 [1<sup>st</sup> Translation Theorem]:** If  $\mathcal{L}\{F(t)\} = F(s)$  &  $a \in \mathbb{R}$ , then

i.e.  $\mathcal{L}\{F(s-a)\} = \mathcal{L}\{F(s)|_{s \rightarrow s-a}\} = e^{at} F(t)$ .

$\mathcal{L}\{e^{at} F(t)\} = F(s-a)$ .

**Proof:**  $\mathcal{L}\{e^{at} F(t)\} = \int_0^{\infty} e^{-st} e^{at} F(t) dt = \int_0^{\infty} e^{-(s-a)t} F(t) dt = F(s-a)$ .



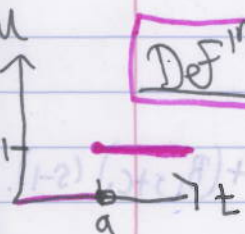
e.g.  $\mathcal{L}\{t e^{-bt}\} = \mathcal{L}\{t\} |_{s \rightarrow s+b} = \frac{1}{(s+b)^2}$ .

used a lot in engineering (esp. electrical circuits)

$\mathcal{L}\{t\} = \frac{1}{s^2}$

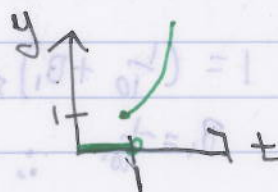
**Def<sup>n</sup>:** The unit step function (Heaviside Function)  $u(t-a)$  is defined to be

$$u(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases}$$



When a function  $F$  defined for  $t \geq 0$  is multiplied by  $u(t-a)$ , the unit step function "turns off" a portion of the graph of  $F$ .

e.g.  $F(t) = t^2 u(t-1)$  has graph:





• Model piece-wise functions using  $u(t-a)$ :

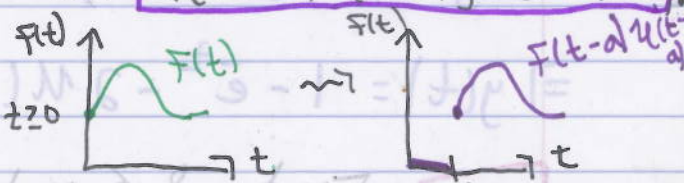
$$f(t) = \begin{cases} 0 & 0 \leq t < a \\ g(t) & a \leq t < b \\ 0 & t \geq b \end{cases}$$

can be written as:

$$f(t) = g(t) [u(t-a) - u(t-b)]$$

**Theorem 7.3.2** [2<sup>nd</sup> Translation Theorem]: IF  $F(s) = \mathcal{L}\{f(t)\}$  &  $a > 0$ , then  $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s)$ .

Proof: [see pg. 294]



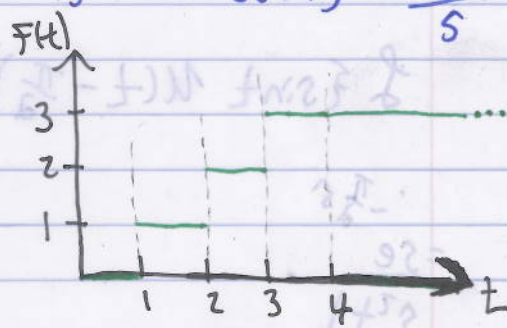
e.g.7 Find  $\mathcal{L}\{u(t-a)\}$ .

Take  $f(t) = 1 \Rightarrow f(t-a) = 1 \Rightarrow \mathcal{L}\{u(t-a)\} = e^{-as} \mathcal{L}\{1\} = \frac{e^{-as}}{s}$ .

e.g.7 Find the Laplace transform of

$$f(t) = u(t-1) + u(t-2) + u(t-3)$$

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}$$



**Alternative Form of Theorem 7.3.2:**  $\mathcal{L}\{g(t)u(t-a)\} = e^{-as} \mathcal{L}\{g(t+a)\}$

**Inverse Form:**  $\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)u(t-a)$

e.g.7  $y' + y = f(t)$ ,  $y(0) = 0$ , where  $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ -1 & t \geq 1 \end{cases}$

$$f(t) = 1 - 2u(t-1)$$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{1 - 2u(t-1)\}$$

$$[sY(s) - y(0)] + Y(s) = \frac{1}{s} - 2\frac{e^{-s}}{s}$$



$$\Rightarrow Y(s) = \frac{1}{s(s+1)} - 2 \frac{e^{-s}}{s(s+1)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s(s+1)} \right\} - 2 \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s(s+1)} \right\}$$

$$\frac{1}{s(s+1)} = \frac{A_1}{s} + \frac{A_2}{s+1} \Rightarrow 1 = A_1(s+1) + A_2 s \Rightarrow A_1 = 1, A_2 = -1.$$

$$\Rightarrow y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - 2 \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s} \right\} + 2 \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s+1} \right\}$$

$$\Rightarrow y(t) = t - e^{-t} - 2u(t-1) + 2e^{-(t-1)}u(t-1).$$

**e.g.7** Find  $\mathcal{L} \left\{ \sin t u\left(t - \frac{\pi}{2}\right) \right\}$ .

$$\mathcal{L} \left\{ \sin t u\left(t - \frac{\pi}{2}\right) \right\} = e^{-\frac{\pi}{2}s} \mathcal{L} \left\{ \sin\left(t + \frac{\pi}{2}\right) \right\} = e^{-\frac{\pi}{2}s} \mathcal{L} \left\{ \cos t \right\}$$

$$\frac{-\frac{\pi}{2}s}{s^2+1}$$

**e.g.7** Solve  $2y'' + 20y' + 51y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 0$ .

$$2[s^2 Y(s) - s y(0) - y'(0)] + 20[s Y(s) - y(0)] + 51 Y(s) = 0$$

$$\Rightarrow Y(s) [2s^2 + 20s + 51] = 4s + 40$$

$$\Rightarrow y(t) = \mathcal{L}^{-1} \left\{ \frac{2s+20}{s^2+10s+\frac{51}{2}} \right\} = \mathcal{L}^{-1} \left\{ \frac{2s+20}{(s+5)^2 + \frac{1}{2}} \right\} = 2 \mathcal{L}^{-1} \left\{ \frac{s+5}{(s+5)^2 + \frac{1}{2}} \right\}$$

$$+ 2 \mathcal{L}^{-1} \left\{ \frac{5}{(s+5)^2 + \frac{1}{2}} \right\} = 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + \frac{1}{2}} \mid s \rightarrow s+5 \right\}$$

1st - Translation Theorem

Not have constant coefficients  
 Laplace transform can solve DEs  
 Laplace transform can solve DEs

$$+ a e^{-t} \left\{ \frac{5}{(s+2)^2} \mid s=5+5 \right\} = a e^{-5t} \left[ \cos\left(\frac{t}{\sqrt{a}}\right) + \sqrt{a} \sin\left(\frac{t}{\sqrt{a}}\right) \right]$$

Integration by Parts

When  $f(x) = u(x)v(x)$  is the product of two functions,  $u$  and  $v$ , then  
 $\int u(x)v(x) dx = u(x) \int v(x) dx - \int u'(x) \int v(x) dx$

When  $f(x) = u(x)v(x)$  is the product of two functions,  $u$  and  $v$ , then  
 choose  $u$  and  $v$  such that  $u$  is easy to differentiate and  $v$  is easy to integrate

Let  $f(x) = x e^{-x}$

$$= \int x e^{-x} dx = x \int e^{-x} dx - \int x' \int e^{-x} dx$$

$$= -x e^{-x} - \int -e^{-x} dx = -x e^{-x} + e^{-x} + C$$

Let  $f(x) = x^2 e^{-x}$

$$= \int x^2 e^{-x} dx = x^2 \int e^{-x} dx - \int 2x \int e^{-x} dx$$

$$= -x^2 e^{-x} - 2 \int x e^{-x} dx = -x^2 e^{-x} - 2(-x e^{-x} + e^{-x}) + C$$

$$= -x^2 e^{-x} + 2x e^{-x} - 2e^{-x} + C$$

Let  $f(x) = x^3 e^{-x}$

$$= \int x^3 e^{-x} dx = x^3 \int e^{-x} dx - \int 3x^2 \int e^{-x} dx$$

$$= -x^3 e^{-x} - 3 \int x^2 e^{-x} dx = -x^3 e^{-x} - 3(-x^2 e^{-x} + 2x e^{-x} - 2e^{-x}) + C$$

$$= -x^3 e^{-x} + 3x^2 e^{-x} - 6x e^{-x} + 6e^{-x} + C$$

Let  $f(x) = x^4 e^{-x}$

$$= \int x^4 e^{-x} dx = x^4 \int e^{-x} dx - \int 4x^3 \int e^{-x} dx$$

$$= -x^4 e^{-x} - 4 \int x^3 e^{-x} dx = -x^4 e^{-x} - 4(-x^3 e^{-x} + 3x^2 e^{-x} - 6x e^{-x} + 6e^{-x}) + C$$

$$= -x^4 e^{-x} + 4x^3 e^{-x} - 12x^2 e^{-x} + 24x e^{-x} - 24e^{-x} + C$$