4 $y(t) = \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$= \frac{1}{10}e^{t} - \frac{1}{10}\cos(3t) - \frac{1}{30}\sin(3t)$.

Math 2C03 - Class # 7

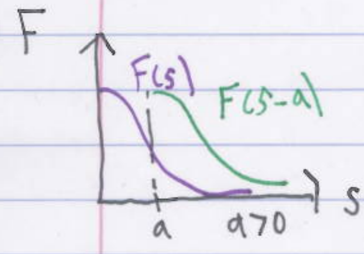
7.3: Operational Properties I

In this section we'll explore several properties of $\mathcal{L}$ to build a more extensive list of transforms.

Theorem 7.3.1 [1st Translation Theorem] [s-axis]: If $\mathcal{L}\{f(t)\} = F(s)$ & $a \in \mathbb{R}$, then $\boxed{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$.

i.e. $\boxed{\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t)}$.

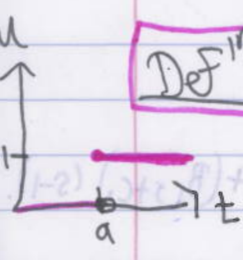
Proof: $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$.



e.g. $\mathcal{L}\{te^{-6t}\} = \mathcal{L}\{t\}|_{s\to s+6} = \frac{1}{(s+6)^2}$.

$\mathcal{L}\{t\} = 1/s^2$

used a lot in engineering (e.g. electrical circuits)

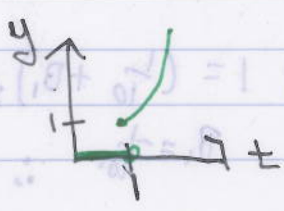


Def'n: The unit step function (Heaviside function) $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}.$$

When a function f defined for $t \ge 0$ is multiplied by $\mathcal{U}(t-a)$, the unit step function "turns off" a portion of the graph of f.

e.g. $f(t) = t^2\,\mathcal{U}(t-1)$ has graph:



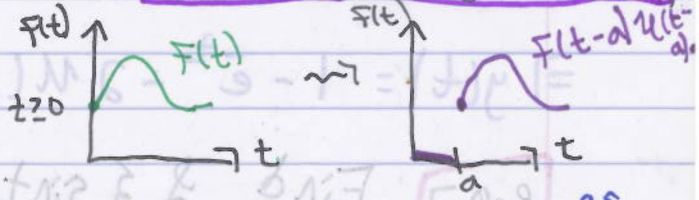
• Model piece-wise functions using $\mathcal{U}(t-a)$:

$F(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0 & t \ge b \end{cases}$ can be written as:

$F(t) = g(t)\left[\mathcal{U}(t-a) - \mathcal{U}(t-b)\right].$

$\boxed{\text{Theorem 7.3.2}}$ [$\underline{2^{nd} \text{ Translation Theorem}}$]: If $F(s) = \mathcal{L}\{F(t)\}$ & $a > 0$,
[$\underline{t\text{-axis}}$]
then $\boxed{\mathcal{L}\{F(t-a)\,\mathcal{U}(t-a)\} = e^{-as} F(s)}$.

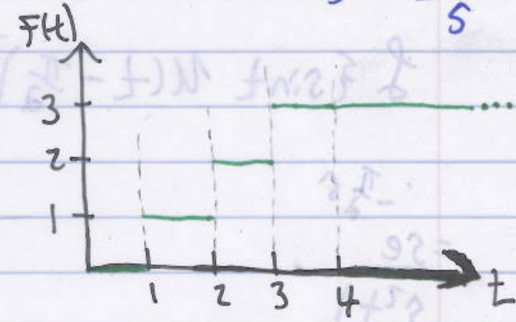
Proof: [see pg. 294]



e.g. → Find $\mathcal{L}\{\mathcal{U}(t-a)\}$.

Take $F(t) = 1 \Rightarrow F(t-a) = 1 \Rightarrow \mathcal{L}\{\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{1\} = \frac{e^{-as}}{s}$.

$\boxed{\text{e.g.}\rightarrow}$ Find the Laplace transform of



$F(t) = \mathcal{U}(t-1) + \mathcal{U}(t-2) + \mathcal{U}(t-3)$.

$\mathcal{L}(F(t)) = \frac{e^{-s} + e^{-2s} + e^{-3s}}{s}$.

$\boxed{\text{Alternative Form of Theorem 7.3.2:}}$ $\boxed{\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}}$.

$\boxed{\text{Inverse Form:}}$ $\boxed{\mathcal{L}^{-1}\{e^{-as} F(s)\} = F(t-a)\,\mathcal{U}(t-a)}$.

$\boxed{\text{e.g.}\rightarrow}$ $y' + y = F(t)$, $y(0) = 0$, where $F(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & t \ge 1 \end{cases}$.

$F(t) = 1 - 2\,\mathcal{U}(t-1)$.

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{1 - 2\,\mathcal{U}(t-1)\}$

$[sY(s) - y(0)] + Y[s] = \frac{1}{s} - 2\frac{e^{-s}}{s}$

$$\Rightarrow Y(s) = \frac{1}{s(s+1)} - 2\frac{e^{-s}}{s(s+1)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{e^{-5}}{s(s+1)}\right\}$$

$$\frac{1}{s(s+1)} = \frac{A_1}{s} + \frac{A_2}{(s+1)} \Rightarrow 1 = A_1 s + A_1 + A_2 s \Rightarrow A_1 = 1,\ A_2 = -1.$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+1}\right\}$$

$$\Rightarrow y(t) = 1 - e^{-t} - 2\,\mathcal{U}(t-1) + 2e^{-(t-1)}\,\mathcal{U}(t-1).$$

$\boxed{\text{e.g.7}}$ Find $\mathcal{L}\{\underbrace{\sin t}_{g(t)}\,\mathcal{U}(t - \underbrace{\pi/2}_{a})\}$.

$$\mathcal{L}\{\sin t\ \mathcal{U}(t-\tfrac{\pi}{2})\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{\underbrace{\sin(t+\tfrac{\pi}{2})}_{\cos t}\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos t\}$$

$$= \frac{s e^{-\frac{\pi}{2}s}}{s^2+1}$$

$\boxed{\text{e.g.7}}$ Solve $2y'' + 20y' + 51y = 0,\quad y(0) = 2,\ y'(0) = 0.$

$$2[s^2 Y(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{0}] + 20[sY(s) - \underbrace{y(0)}_{2}] \quad 51Y(s) = 0$$

$$\Rightarrow Y(s)[2s^2 + 20s + 51] = 4s + 40$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{2s+20}{s^2+10s+\frac{51}{2}}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+20}{(s+5)^2+\frac{1}{2}}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s+5}{(s+5)^2+\frac{1}{2}}\right\}$$

$$+ 2\mathcal{L}^{-1}\left\{\frac{5}{(s+5)^2+\frac{1}{2}}\right\} = 2\mathcal{L}^{-1}\left\{\frac{5}{s^2+\frac{1}{2}}\Big|_{s\to s+5}\right\}$$

1st - Translation Theorem

$$+\,2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{1}{2}}\Bigg|_{s\to s+5}\right\} = 2e^{-5t}\left[\cos\left(\frac{t}{\sqrt{2}}\right) + \sqrt{2}\,5\sin\left(\frac{t}{\sqrt{2}}\right)\right].$$