Math 2C03 - Class #6

Midterm Info: Wed. July 15^{th}, 7pm - 8:15pm.

- HH 104: Basinskifer is - Or.
- BSB 105: Peng - Zhu.

- Lecture in BSB105 from 8:30pm - 9:45pm.

Format:
- 3 Fill in the blank
- 6 Full answer questions
- Approx. 85% WeBWork type questions
  [e.g. solve the following DE...].

Half of the battle is recognizing the type of DE:

1^{st} - order

Yes → [2.2, 2.3, 2.4] Separable? Linear? Exact? — Yes → solve

No → [2.5] Bernoulli? Homogeneous? $F(Ax+By+C)=0$?

Yes → Make substitution & solve the corr. separable / linear eq^n

No → [2.4] Can I find an integrating factor to make it exact?

Yes → solve corr. exact eq^n

No → must have made a mistake somewhere!

1^{st} - order: No → It must be linear. Constant coefficients?

Yes → homogeneous?

Yes → [2.3] Solve as in 2.3.

No → $g(x)$ sum/product of poly.'s, expon.'s, sines, & cos's?

Yes → [4.5] Annihilator Approach

No → [4.6] Variation of Parameters

Constant coefficients? No → [4.7] Cauchy-Euler?

Yes → [4.6 & 4.7] Solve. If non-homog., use variation of parameters too.

No → [4.2] Must know a solution y_1 already & use Reduction of order

Ch. 7: The Laplace Transform

<u>Motivation:</u> Laplace Transforms allow us to solve linear differential DE's w/ constant coefficients:
$a_n y^{(n)} + \cdots + a_0 y = g(x)$.
It's especially useful when $g(x)$ has a jump discontinuity (advantage over techniques learned in Ch. 4).
These types of DE's model phenomena in many real-word applications (electrical circuits, probability theory).

7.1: Def^{n} of Laplace Transform:

Def^{n}: Let F be a function defined for $t \geq 0$. Then the <u>Laplace transform</u> of F is the integral

$$F(s) := \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt,$$ provided that the integral converges.

e.g. Evaluate $\mathcal{L}\{1\}$.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt = \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b$$

$$= \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s} \text{ if } s>0.$$

e.g. Evaluate $\mathcal{L}\{t\}$.

$u=t$
$du=dt$
$v=-\frac{e^{-st}}{s}$
$dv=e^{-st}$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \left.-\frac{te^{-st}}{s}\right|_0^\infty + \underbrace{\frac{1}{s}\int_0^\infty e^{-st}\,dt}_{\frac{1}{s}}$$

$$= \frac{1}{s^2}.$$

$\boxed{\text{e.g.}}$ Evaluate $\mathcal{L}\{e^{5t}\}$.

$$\mathcal{L}\{e^{5t}\} = \int_0^\infty e^{-st} e^{5t}\, dt = \int_0^\infty e^{(5-s)t}\, dt = \frac{1}{5-s} e^{(5-s)t} \Big|_0^\infty$$

$$= -\frac{1}{5-s} = \frac{1}{s-5}.$$

if $5-s<0$

* We don't always write the restrictions on s. It's understood that s is sufficiently restricted to guarantee convergence. *

Transforms of some Basic Functions:

a) $\mathcal{L}\{1\} = 1/s$.

b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $n = 1, 2, 3, \dots$

c) $\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$.

d) $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$

e) $\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$

f) $\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$

g) $\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$.

Exercise: Choose some small values of n, α, & k & compute these directly.

• $\mathcal{L}$ is a linear transform:

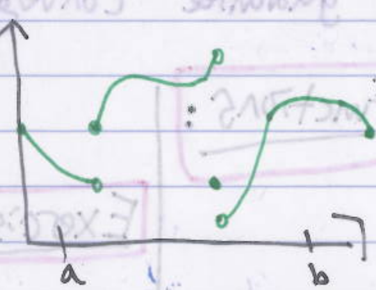
$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}.$$

* this follows from the linearity of the integral $\int_0^\infty$. *

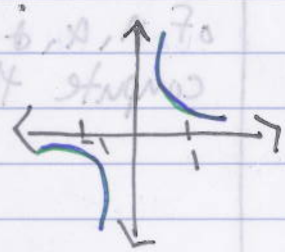
Def^n: • A function f is <u>piecewise continuous</u> on an interval $[a,b]$ if $[a,b]$ can be subdivided into a finite number of subintervals, & in each subinterval $f(t)$ is continuous & has a finite left & right limit.

i.e.) It has a finite number of breaks & doesn't blow up to infinity anywhere. (no vertical asymptotes, only jump discontinuities).

e.g.)



is piecewise cont. on $[a,b]$.



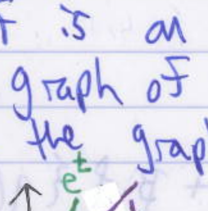
$y = 1/x$ ... <u>not</u> piecewise cont. on $[-1,1]$ b/c left & right limits approach ∞ & $-\infty$.

• $f(t)$ piecewise cont. on $[0,\infty)$ if piecewise cont. on $[0,b]$ for every $b > 0$.

Def^n: A function f is of <u>exponential order c</u> if there exist constants c, $M > 0$, $T > 0$ s.t. $|f(t)| \leq Me^{ct}$ $\forall\, t > T$.

i.e.) If $\lim_{t \to \infty} \frac{|f(t)|}{e^{ct}} = L$, where $L \geq 0$ is finite.

e.g.) If f is an increasing function, this says that the graph of f on (T,∞) does not grow faster than the graph of Me^{ct}, where c positive constant.



Theorem 7.1.2: Existence of the Laplace Transform [sufficient conditions]

If F is piecewise continuous on $[0,\infty)$ & of exponential order c, then $\mathcal{L}\{F(t)\}$ exists for $s > c$.

Proof [see pg. 278]

7.2: Inverse Transforms & Transforms of Derivatives

Def^{n}: Given a function $F(s)$, if there is a function $g(t)$ that is continuous on $[0,\infty)$ & satisfies $\mathcal{L}\{g(t)\} = F(s)$, then $g(t)$ is the inverse Laplace transform of $F(s)$ & we write $g(t) = \mathcal{L}^{-1}\{F(s)\}$.

e.g.) $\mathcal{L}^{-1}\{1/s\} = 1$, $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$, etc.

e.g.) Find $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$.

We know $\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3 \Rightarrow \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} \overset{\mathcal{L}^{-1}\text{ linear}}{=} \frac{1}{3!}t^3 = \frac{t^3}{6}$.

Theorem: $\mathcal{L}^{-1}$ is a linear transform.

i.e.) $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$.

Proof: Suppose $F(s) = \mathcal{L}\{F(t)\}$ & $G(s) = \mathcal{L}(g(t))$. So, $\mathcal{L}^{-1}\{F(s)\} = F(t)$ & $\mathcal{L}^{-1}\{G(s)\} = g(t)$.

$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \mathcal{L}^{-1}\{\alpha\mathcal{L}\{F(t)\} + \beta\mathcal{L}\{g(t)\}\} \overset{\mathcal{L}\text{ linear}}{=} \mathcal{L}^{-1}\{\mathcal{L}\{\alpha F(t) + \beta g(t)\}\}$

$\overset{\mathcal{L}^{-1}\circ\mathcal{L} = ID}{=} \alpha F(t) + \beta g(t) = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{g(s)\}$.

Review: Partial Fractions: consider a rational function $\frac{P(s)}{Q(s)}$ where $P(s)$ & $Q(s)$ are polynomials with real coefficients & $\deg(P(s)) < \deg(Q(s))$.

$\boxed{1}$ Factor & cancel common factors of $P(s)$ & $Q(s)$.

$\boxed{2}$ For each linear term $(s-a)^m$, $a \in \mathbb{R}$ in the denominator, include terms of the form:

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_m}{(s-a)^m}.$$

$\boxed{3}$ For each irreducible quadratic term $[(s-\alpha)^2 + \beta^2]^p$, $\alpha, \beta \in \mathbb{R}$, in denom. include terms of the form

$$\frac{B_1 s + C_1}{(s-\alpha)^2 + \beta^2} + \frac{B_2 s + C_2}{((s-\alpha)^2+\beta^2)^2} + \dots + \frac{B_p s + C_p}{((s-\alpha)^2+\beta^2)^p}.$$

$\boxed{4}$ Set $\frac{P(s)}{Q(s)}$ equal to the sum of these terms.

$\boxed{5}$ Put over common denominator.

$\boxed{6}$ Equate numerators.

$\boxed{7}$ (a) Find A_i, B_i, C_i by equating coefficients s^k

or

(b) evaluate both sides at the roots.

e.g.7 Find $\mathcal{L}^{-1}\left\{\frac{s}{(s+2)(s^2+4)}\right\}$.

$$\frac{s}{(s+2)(s^2+4)} = \frac{A_1}{s+2} + \frac{B_1 s + C_1}{(s^2+4)} = \frac{A_1(s^2+4) + (B_1 s + C_1)(s+2)}{(s+2)(s^2+4)}$$

(*) $\Rightarrow s = A_1 s^2 + 4A_1 + B_1 s^2 + 2B_1 s + C_1 s + 2C_1$

$\Rightarrow s = (A_1+B_1)s^2 + (2B_1 + C_1)s + (4A_1 + 2C_1)$

$\Rightarrow A_1 + B_1 = 0$ & $2B_1 + C_1 = 1$ & $4A_1 + 2C_1 = 0$

$\Rightarrow B_1 = \frac{1}{2} - \frac{C_1}{2}$

$\Rightarrow A_1 = -\frac{1}{2} + \frac{C_1}{2}$ $\Rightarrow -2 + 2C_1 + 2C_1 = 0$

$\Rightarrow 4C_1 = 2 \Rightarrow C_1 = \frac{1}{2}$

$\Rightarrow B_1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

So, $\Rightarrow A_1 = -\frac{1}{4}$.

$$\frac{s}{(s+2)(s^2+4)} = \frac{-1}{4(s+2)} + \frac{s}{4(s^2+4)} + \frac{1}{2(s^2+4)}.$$

[Alternatively, at (*) could have evaluated at roots:

$\underline{s=-2}$: $-2 = 8A_1 \Rightarrow A_1 = -\frac{1}{4}$. Then $-\frac{1}{4} + B_1 = 0$ & $2B_1 + C_1 = 1$ & $4(-\frac{1}{4}) + 2C_1 = 0$

$\Rightarrow B_1 = \frac{1}{4}$ & $C_1 = \frac{1}{2}$].

$$\therefore \mathcal{L}^{-1}\left\{\frac{s}{(s+2)(s^2+4)}\right\} = -\frac{1}{4}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}}_{e^{-2t}} + \frac{1}{4}\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}}_{\cos(2t)} + \frac{1}{4}\underbrace{\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}}_{\sin(2t)}$$

$$= -\frac{1}{4}e^{-2t} + \frac{1}{4}\cos(2t) + \frac{1}{4}\sin(2t).$$

Transform of a Derivative:

Theorem 7.2.2: If F is C^{n-1} on $[0,\infty)$ & $F, \dots, F^{(n-1)}$ are of exponential order & if $F^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$\mathcal{L}\{F^{(n)}(t)\} = s^n F(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - F^{(n-1)}(0),$$

where $F(s) = \mathcal{L}\{F(t)\}$.

$u = e^{-st}$
$du = -se^{-st}$
$v = F(t)$
$dv = F'(t)$

e.g. ? $n=1$: $\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st}F'(t)\,dt = e^{-st}F(t)\Big|_0^\infty + \int_0^\infty sF(t)e^{-st}\,dt$

$$= -F(0) + s\underbrace{\int_0^\infty e^{-st}F(t)\,dt}_{\mathcal{L}\{F(t)\}} = -F(0) + sF(s).$$

Solving Linear IVP's w/ Constant Coef. using Laplace Transforms:

* Although ch.4 provides methods for doing this too, often IVP's are easier to solve using Laplace transforms. *

$$\begin{cases} a_n y^{(n)} + \dots + a_0 y = g(t) \\ y(0) = y_0;\ y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1}. \end{cases}$$

$\boxed{1}$ Apply $\mathcal{L}$ to both sides:

$$a_n \mathcal{L}\{y^{(n)}\} + \dots + a_0\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}.$$

$\boxed{2}$ By Theorem 7.2.2, this eq'n becomes:

$$a_n[s^n Y(s) - s^{n-1}y(0) - \dots - y^{(n-1)}(0)]$$
$$+ a_{n-1}[s^{n-1}Y(s) - s^{n-2}y(0) - \dots - y^{(n-2)}(0)] + \dots + a_0 Y(s) = G(s).$$

$\boxed{3}$ Solve for $Y(s)$.

[4] Apply $\mathcal{L}^{-1}$. This will give you the solution $y(t)$ of the original IVP.

$\mathcal{L}\{0\} = \int_0^\infty 0\,dt = 0.$

e.g.→ $2y' + y = 0$, $y(0) = -3$.

1st-order Linear, so probably easier to solve using our formula from 2.3, but when have higher-order & lots of initial conditions, this Laplace Transforms method becomes easier.

[1] $2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}$

[2] $2[sY(s) - y(0)] + Y(s) = 0$

[3] $2sY(s) + Y(s) = -6$

$Y(s) = \dfrac{-6}{2s+1}$

[4] $\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{-3\left(\dfrac{1}{s+\frac{1}{2}}\right)\right\}$.

$\Rightarrow y(t) = -3e^{-t/2}$.

e.g.→ $y'' + 9y = e^t$, $y(0) = 0$, $y'(0) = 0$.

[1] $\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$

[2] $\left[s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right] + 9Y(s) = \dfrac{1}{s-1}$

[3] $Y(s)[s^2+9] = \dfrac{1}{s-1}$

$Y(s) = \dfrac{1}{(s^2+9)(s-1)} = \dfrac{A_1}{(s-1)} + \dfrac{B_1 s + C_1}{(s^2+9)} \Rightarrow 1 = A_1(s^2+9) + (B_1 s + C_1)(s-1)$.

At $s=1$: $1 = 10A_1 \Rightarrow A_1 = \frac{1}{10}$. $1 = \frac{1}{10}s^2 + \frac{9}{10} + B_1 s^2 - B_1 s + C_1 s - C_1$

$\Rightarrow 1 = \left(\frac{1}{10} + B_1\right)s^2 + (C_1 - B_1)s + \left(\frac{9}{10} - C_1\right) \Rightarrow 1 = \frac{9}{10} - C_1 \Rightarrow C_1 = \frac{-1}{10}$.

$B_1 = -\frac{1}{10}$. $\therefore Y(s) = \dfrac{1}{10(s-1)} + \dfrac{-s}{10(s^2+9)} - \dfrac{1}{10(s^2+9)}$.

4 $y(t) = \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$= \frac{1}{10}e^{t} - \frac{1}{10}\cos(3t) - \frac{1}{30}\sin(3t).$