4.5: Undetermined Coefficients - Annihilator Approach

- In this section, we'll learn how to solve certain types of nonhomogeneous DE's with constant coefficients:

$$(*)\left\{\ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x).\right.$$

For the method in this section to work, we need $g(x)$ to be a function consisting of finite sums & products of constants, polynomials, exponential functions $e^{\alpha x}$, sines, & cosines.

Recall: It's sometimes convenient to write $(*)$ as
$L(y) = g(x)$ or $a_n D^n y + \dots + a_1 D y + a_0 y = g(x)$.

Here L is the linear operator $L = a_n D^n + \dots + a_1 D + a_0$.

e.g.: $3y'' + 15y = x^2 \leftrightarrow (3D^2 + 15)(y) = x^2$.

Def^n: A linear operator L with constant coef. is said to be an annihilator of a sufficiently differentiable function f if $L(f(x)) = 0$.

e.g.: $L = D^2$ annihilates 4 & x, since $\frac{d^2}{dx^2}(4) = \frac{d^2}{dx^2}(x) = 0$.

* Notice: The set of functions that are annihilated by $L = a_n D^n + \dots + a_1 D + a_0$ are those functions which are in the set of solutions of the homog. DE $L(y) = 0$.
i.e.: They can be obtained from the general solution of $L(y) = 0$.

e.g.: Which functions are annihilated by D^n?

To find these we want to consider $D^n(y) = 0$ (i.e.: $y^{(n)} = 0$). The

: aux. eq'n here is $m^n = 0$; so $m=0$ repeated n times $\Rightarrow$ general solution is $c_1 + c_2 x + c_3 x^2 + \dots + c_n x^{n-1}$ $\Rightarrow$ D^n annihilates functions which are linear combinations of $\{1, x, x^2, \dots, x^{n-1}\}$.

e.g. $\Rightarrow$ Which functions are annihilated by $(D-\alpha)^n$?

$(D-\alpha)^n(y) = 0$ has aux. eq'n $(m-\alpha)^n = 0 \Rightarrow \alpha$ root of multiplicity n $\Rightarrow$ general solution is $y = c_1 e^{\alpha x} + c_2 x e^{\alpha x} + \dots + c_n x^{n-1} e^{\alpha x}$
$\Rightarrow (D-\alpha)^n$ annihilates $\text{span}\{e^{\alpha x}, xe^{\alpha x}, \dots, x^{n-1}e^{\alpha x}\}$.

e.g. $\Rightarrow$ Which functions are annihilated by $[D^2 - 2\alpha D + (\alpha^2+\beta^2)]^n$?

$m^2 - 2\alpha m + (\alpha^2+\beta^2) = 0 \Leftrightarrow m = \dfrac{2\alpha \pm \sqrt{4\alpha^2 - 4\alpha^2 - 4\beta^2}}{2} = \alpha \pm i\beta$;
conjugate pair w/ mult. n
$\Rightarrow$ general solution is $y = e^{\alpha x}[c_1 \cos\beta x + \tilde{c}_1 \sin\beta x + c_2 x\cos\beta x + \tilde{c}_2 x \sin\beta x + \dots + c_n x^{n-1}\cos\beta x + \tilde{c}_n x^{n-1}\sin\beta x]$
$\Rightarrow \text{span}\{e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \dots, x^{n-1}e^{\alpha x}\cos\beta x,$
$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \dots, x^{n-1}e^{\alpha x}\sin\beta x\}$ annihilated.

Properties: [1] If y_1 & y_2 are annihilated by L, then $L(y_1+y_2) = 0$.
$[L(y_1+y_2) = L(y_1) + L(y_2) = 0 + 0 = 0]$.

[2] Suppose $L_1(y_1) = 0$, $L_2(y_2) = 0$, but $L_1(y_2) \neq 0$ & $L_2(y_1) \neq 0$.
Then $L_1L_2(y_1+y_2) = 0$.
$[L_1L_2(y_1+y_2) = L_2L_1(y_1) + L_1L_2(y_2) = 0 + 0 = 0]$.

e.g. $\Rightarrow$ Find an operator which annihilates $e^{-x}\sin x - e^{2x}\cos x$.

We know (D^2+2D+2) annihilates $e^{-x}\sin x$, & (D^2-4D+5) annihilates $e^{2x}\cos x$, so $(D^2+2D+2)(D^2-4D+5)$ will annihilate $e^{-x}\sin x - e^{2x}\cos x$.

Method of Undetermined Coefficients - Annihilator Approach:

This is a method for finding the general solution for $L(y) = g(x)$, where $g(x)$ finite sums/products of polynomials, exponential e^{4x}, sines, & cosines.

1. $\boxed{1}$ Find the general solution y_c for homog. eq'n $L(y) = 0$.

2. $\boxed{2}$ Find an operator L_1 which annilates $g(x)$, & operate on both sides of $L(y) = g(x)$ by L_1. [the lowest possible order operator that does the job].

3. $\boxed{3}$ Find general solution for homog. eq'n $L_1 L(y) = 0$.

4. $\boxed{4}$ Delete from solution in $\boxed{3}$ the terms that also appear in y_c in $\boxed{1}$.

5. $\boxed{5}$ Substitute y_p found in $\boxed{4}$ into $L(y) = g(x)$, match coef., & solve for unknown coef. in y_p.

6. $\boxed{6}$ The general solution is $y = y_c + y_p$.

__e.g.7__ Find the general solution for the nonhomog. DE

$$y'' + 3y' = \underbrace{4x - 5}_{g(x)}.$$

$\boxed{1}$ $y'' + 3y' = 0$ has aux. eq'n $m^2 + 3m = 0 \Leftrightarrow m(m+3) = 0 \Leftrightarrow m = 0$ or $m = -3$.

So, general solution is $y_c = c_1 + c_2 e^{-3x}$.

$\boxed{2}$ $L_1 = D^2$ annihilates $4x-5$, & has the lowest possible order [i.e. D^3 & D^4 do the job too, but have higher order].

$\boxed{3}$ Here $L = D^2 + 3D$, so $L_1 L(y) = D^2(D^2 + 3D) = D^4 + 3D^3$.

So, want to find general solution for $y^{(4)} + 3y^{(3)} = 0$.

Aux. eq'n is $\underbrace{m^4 + 3m^3 = 0}_{m^3(m+3)=0} \iff \underbrace{m=0}_{\text{mult. } 3}$ or $\underbrace{m=-3}_{\text{mult. } 1}$.

So, general solution is $c_1 + c_2 x + c_3 x^2 + c_4 e^{-3x}$.

[4] c_1 & $c_4 e^{-3x}$ appear in [1] & [3], so $y_p = c_3 x + c_4 x^2$.

[5] $y_p' = c_3 + 2c_4 x$. $y_p'' = 2c_4$. So, $L(y_p) = g(x)$

$\iff 2c_4 + 3c_3 + 6c_4 x = 4x - 5 \iff (6c_4)x + (2c_4 + 3c_3) = 4x - 5$

$\iff 6c_4 = 4$ & $2c_4 + 3c_3 = -5$

$c_4 = 2/3$ $\quad 4/3 + 3c_3 = -5 \Rightarrow 3c_3 = -\frac{19}{3} \Rightarrow c_3 = -\frac{19}{9}$.

So, $y_p = -\frac{19}{9}x + \frac{2}{3}x^2$.

[6] $\therefore$ The general solution is $y = c_1 + c_2 e^{-3x} + \frac{2}{3}x^2 - \frac{19}{9}x$.

4.6: Variation of Parameters:

[w/ constant coef.]

In section 4.5, we learned to solve $L(y) = g(x)$ where $g(x)$ was the sum/product of poly.'s, $e^{\alpha x}$'s, sines, & cosines using undetermined coefficients. In this section, we'll solve $L(y) = g(x)$ where there is no restricts on $g(x)$.

* Variation of Parameters can also be used to solve $L(y) = g(x)$ where L doesn't have constant coefficients [see #23, 32 in 4.6 & see section 4.7]. *

Variation of Parameters for $n=2$: [$a_2 y'' + a_1 y' + a_0 y = g(x)$]

[1] Find general solution $y_c = c_1 y_1 + c_2 y_2$ for homog. $a_2 y'' + a_1 y' + a_0 y = 0$.

[2] Compute Wronskian $W(y_1(x), y_2(x))$.

3. Put eq^n in standard form $y'' + Py' + Qy = f(x)$.

4. Compute $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$.

5. Find $u_1 := \int \frac{W_1}{W}\,dx$ & $u_2 := \int \frac{W_2}{W}\,dx$.

6. A particular solution is $y_p = u_1 y_1 + u_2 y_2$ & the general solution is $y = y_c + y_p$.

Idea Behind Solution Method: Want to find a particular solution of the form $y = u_1(x)y_1(x) + u_2(x)y_2(x)$.

[Plug this into the original DE, follow your nose, solve for u_1' & u_2'].

---

e.g. 7 Find the general solution of $y'' + y = \tan x$.

1. $y'' + y = 0$ has aux. eq^n $m^2 + 1 = 0 \Rightarrow m = \pm i$. $\alpha = 0, \beta = 1$.

So, $y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$ general solution to homog. eq^n.

2. $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$

3. $\underbrace{y}_{Q}{}'' + y = \underbrace{\tan x}_{f(x)}$.

4. $W_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = -\sin x \tan x$. $W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \sin x$.

5 $u_1 = \int \frac{W_1}{W}\, dx = -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1}{\cos x} - \cos x \; dx$

$= -\ln|\sec x + \tan x| + \sin x + c.$

$u_2 = \int \sin x \, dx = -\cos x + c.$

6 So, $y_p = \sin x \cos x - \ln|\sec x + \tan x|\cos x - \cos x \sin x$
$= -\ln|\sec x + \tan x|\cos x$ is a particular solution.

$y = c_1 \cos x + c_2 \sin x - \ln|\sec x + \tan x|\cos x$ general solution.

Variation of Parameters for General n case:

The method shown in the $n=2$ case naturally generalizes.
$y_c = c_1 y_1 + \dots + c_n y_n$.
Here $u_k' = \frac{W_k}{W}$, where W_k is the det. of the matrix

obtained by replacing the k^{th} column of the Wronskian by
$(0 \; 0 \cdots 0 \; f(x))^T$, & $W = W(y_1, \dots, y_n)$.

$y_p = u_1 y_1 + \dots + u_n y_n$. General solution is $y = y_c + y_p$.

4.7: Cauchy-Euler Eq'n

In 4.3-4.6 we solved linear DE's w/ constant coef..
In this section, we'll learn how to solve a special type of linear DE w/ non-constant coef..

Def^n: A linear DE of the form $a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = g(x)$
is known as a Cauchy-Euler eq'n.
[Here the a_i are constants].

We solve these DE's in a manner very similar to sections 4.3 [Homog. w/ constant coef.] & 4.6 [Variation of parameters].

Method of Solution:

1 Solve the homog. eq'n $a_n x^n y^{(n)} + \dots + a_0 y = 0$ by trying a solution of the form $y = x^m$. [in 4.3 we did this, but instead used $y = e^{mx}$]. Plug $y = x^m$ into the eq'n. Each term $a_k x^k y^{(k)}$ will become $a_k x^m m(m-1)(m-2)\cdot \dots \cdot (m-k+1)$.

2 We seek a solution on $(0,\infty)$, so factor x^m out of the above eq'n. $y = x^m$ is a solution of the DE whenever m is a solution to this auxiliary eq'n.

3 Analogous to 4.3, if we have the following cases:

a If we have distinct roots $m_1, \dots, m_k$, the general solution will contain the linear combination $c_1 x^{m_1} + \dots + c_k x^{m_k}$.

b If we have a root m_1 of multiplicity k, then the general solution will contain the linear combination $c_1 x^{m_1} + c_2 x^{m_1} \ln x + c_3 x^{m_1} (\ln x)^2 + \dots + c_k x^{m_1} (\ln x)^{k-1}$.

c If we have a complex conjugate pair $\alpha \pm \beta i$, then the general solution will contain the linear combination $x^{\alpha} [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$.

4 Use variation of parameters to solve the nonhomog. Cauchy-Euler eq'n. [y_c was obtained in steps 1-3].

e.g. 7 Solve $2x^2y'' + 5xy' + y = x^2 - x$.

m^2+3m+2

$(m+2)(m+1)$

$(2m+2)(2m+1)$

$(m+1)(2m+1)$

$\boxed{1}$ $y = x^m \Rightarrow 2x^m(m(m-1)) + 5x^m m + x^m = 0$

on $(0,\infty)$ $\Rightarrow x^m[2m^2 + 3m + 1] = 0$

$\boxed{2}$ $\Rightarrow (2m+1)(m+1) = 0 \Rightarrow m = -1$ or $m = -\frac{1}{2}$.

$\boxed{3}$ So, $y_c = c_1 \underbrace{x^{-1}}_{y_1} + c_2 \underbrace{x^{-1/2}}_{y_2}$.

$\boxed{4}$ $W = \begin{vmatrix} x^{-1} & x^{-1/2} \\ -x^{-2} & -\frac{1}{2}x^{-3/2} \end{vmatrix} = -\frac{1}{2}x^{-5/2} + x^{-5/2} = \frac{1}{2}x^{-5/2}$.

$x^2y'' + \frac{5}{2}x^{-1}y' + \frac{1}{2}x^{-2}y = \underbrace{\frac{1}{2} - \frac{1}{2}x^{-1}}_{F(x)}$.

$W_1 = \begin{vmatrix} 0 & x^{-1/2} \\ \frac{1}{2} - \frac{1}{2}x^{-1} & -\frac{1}{2}x^{-3/2} \end{vmatrix} = -\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2}$.

$u_1 = \int -x^2 + x \, dx = -\frac{1}{3}x^3 + \frac{1}{2}x^2$.

$W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{1}{2} - \frac{1}{2}x^{-1} \end{vmatrix} = \frac{1}{2}x^{-1} - \frac{1}{2}x^{-2}$.

$u_2 = \int x^{3/2} - x^{1/2} \, dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$.

$y = c_1x^{-1} + c_2x^{-1/2} - \frac{1}{3}x^2 + \frac{1}{2}x + \frac{2}{5}x^2 - \frac{2}{3}x$

$= c_1x^{-1} + c_2x^{-1/2} + \frac{1}{15}x^2 - \frac{1}{6}x$.

e.g. 7 Solve $x^4y^{(4)} + 6x^3y^{(3)} + 9x^2y'' + 3xy' + y = 0$.

$y = x^m \Rightarrow x^m[m(m-1)(m-2)(m-3) + 6(m)(m-1)(m-2) + 9m(m-1) + 3m + 1] = 0$

$\Rightarrow (m^2-m)(m^2-5m+6) + 6(m^2-m)(m-2) + 9m^2 - 9m + 3m + 1 = 0$

$\Rightarrow m^4 - \cancel{6m^3} + 11m^2 - \cancel{6m} + \cancel{6m^3} - 18m^2 + \cancel{12m} + 9m^2 - \cancel{6m} + 1 = 0$

$\Rightarrow m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2+1)^2 = 0 \Rightarrow \underbrace{m = \pm i}$ $\alpha=0, \beta=1$ double pair

So, the general solution is:

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + c_3 \ln x \cos(\ln x) + c_4 \ln x \sin(\ln x).$$