Math 2C03 - Class # 4

Mon. July 6th, 2015

Math 2C03: Quiz #2

MONDAY, JULY 6TH, 7PM (FIRST 10 MINUTES OF CLASS)
McMaster University

Name: *marking scheme* Student ID: ____________

Please answer each question fully, providing all reasoning

Questions:

1. In class last Monday we discussed three types of differential equations: homogeneous, Bernoulli's equation, and DE's of the form $y' = f(Ax + By + C)$, for $A, B, C \in \mathbb{R}$, $B \neq 0$. These equations can all be solved by making an appropriate substitution which transforms them into a separable or linear equation.
   (a) (3pts) Define each type of equation.
   (b) (3pts) In each case, what substitution should be made to solve it?
   (c) (3pts) For each, which type of equation does the differential equation become after making that substitution?
   (d) (3pts) Give an example of each type of equation.

(a) (1pt) [i] A 1st-order DE $M(x,y)dx + N(x,y)dy = 0$ is homogeneous if M & N are homogeneous functions of the same degree. i.e. $M(tx,ty) = t^\alpha M(x,y)$ & $N(tx,ty) = t^\alpha N(x,y)$ for some $\alpha \in \mathbb{R}$.

(1pt) [ii] Bernoulli's eqn is a DE of the form $y' + P(x)y = f(x)y^n$.

(1pt) [iii] $y' = f(Ax+By+C)$ is a 1st-order DE $y' = g(x,y)$, where $g(x,y)$ can be written as a function of $u = Ax+By+C$.
→ [For this one, the name $y' = f(Ax+By+C)$ is kind of self-explanatory... so even if they say it's a DE of the form $y' = f(Ax+By+C)$, that would be fine].

(b) (1pt) [i] Make the substitution $y = ux$ or $x = vy$.
(1pt) [ii] $u = y^{1-n}$
(1pt) [iii] $u = Ax + By + C$

(c) (1pt) [1] ⤳ separable

(1pt) [5] ⤳ linear

(1pt) [6] ⤳ separable

(d) (1pt) [1] $\frac{dy}{dx} = \frac{x^2 - y^2}{3xy} \iff 3xy\,dy + (-x^2 + y^2)\,dx$ homog. of degree 2.

(1pt) [4] $y' - 5y = \frac{-5}{2} x y^3$ Bernoulli's eq^n, with $n = 3$.

(1pt) [6] $y' = (-2x + y)^2 - 7$ has the form $y' = F(Ax + By + c)$, where $A = -2$, $B = 1$, $c = 0$.

* any examples that work are fine here.*

Reminders:

- Solutions to the assigned Practice Problems in Chapters 1 and 2 are posted. Please be sure to try these questions on your own before viewing the solutions... it’s hard to digest how an algorithm works without trying it yourself!

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- Solutions to the assigned Practice Problems in Chapters 1 and 2 are posted. Please be sure to try these questions on your own before viewing the solutions... it's hard to digest how an algorithm works without trying it yourself!
- Your third assignment is due on Friday.
  - The written portion should be deposited in the assignment locker (C33) located in the basement of Hamilton Hall by 2pm on Friday. If you want to submit via email, please type of scan it, make sure your file is a PDF, and title it LastName_FirstName_Assignment3.
  - The online WeBWork portion must be completed by 11:59pm Friday.

Assignment #2:

- 1. Consider the initial value problem $2y' + 8xy = x^3 e^{x^2}$, $y(0) = 2$. Without solving this IVP, explain why a solution exists. Can there exist more than one solution to this IVP on a given interval? Explain.

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- **Existence of a Unique Solution (1st-Order IVP's):** Let $R = [a,b] \times [c,d]$ contain the point (x_0, y_0) in its interior. If $f(x,y)$ and $\frac{df}{dy}$ are continuous on R, then there exists some interval I_0 containing x_0 contained in $[a,b]$ and a unique function $y(x_0)$ defined on I_0 such that $y(x)$ is a unique solution to the IVP $y' = f(x,y), y(x_0) = y_0$.

- **Existence of a Unique Solution (*Linear* 1st-Order IVP's):** Consider the IVP $y' + P(x)y = f(x)$, $y(x_0) = y_0$. If $P(x)$ and $f(x)$ are continuous on an interval I containing x_0, then there exists a unique solution of this IVP on I.

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- **Notice:** In the first theorem we're only guaranteed uniqueness on some interval I_0, whereas in the second we're guaranteed uniqueness on the entire interval I where $P(x)$ and $f(x)$ are unique! Therefore, in this question, we need the LINEAR theorem, because this will guarantee uniqueness on $(-\infty, \infty)$, and so there will be a unique solution on ANY interval. (i.e. There can't exist more than one solution on a given interval).

Assignment #2:

- 3. Suppose you are given a first-order differential equation $y' = f(x,y)$, which satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R. Could two different solution curves in its 1-parameter family of solutions intersect at a point in R? Why or why not?

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- ***state Theorem 1.2.1 verbatim****. Therefore by Theorem 1.2.1 no two solutions can intersect at a point.
- Consider an arbitrary point (x_0, y_0) in R. The hypotheses of Theorem 1.2.1 are satisfied on R, taking $y(x_0) = y_0$ as an initial condition. Therefore, we know that there exists an interval I_0 containing the point (x_0, y_0) such that a solution $y(x)$ exists and is unique. Therefore, no other solution curve, distinct from $y(x)$, can go through the point (x_0, y_0), because it would have to pass through the interval I_0, and we know there is only one unique solution curve in I_0 passing through (x_0, y_0). The point (x_0, y_0) was chosen arbitrarily, so this is true of all points in R. Therefore, no two solution curves can intersect at a point in R.

Math 2C03 - Assignment #2 [Written Part]

[4pts] 1. Consider the IVP $2y' + 8xy = x^3 e^{x^2}$, $y(0) = 2$. Without solving this IVP, explain why a solution exists. Can there exist more than one solution to this IVP on a given interval? Explain.

[From class] Recall: Theorem [Existence & Uniqueness of 1st-order Linear IVPs]

Consider the IVP $y' + P(x)y = F(x)$, $y(x_0) = y_0$. If $P(x)$ & $F(x)$ are cont. on an interval I containing x_0, then $\exists!$ solution of the IVP on I.

Here, our eq'n is linear 1st-order: $y' + \underbrace{4x}_{P(x)} y = \underbrace{\frac{1}{2}x^3 e^{x^2}}_{F(x)}$.

$P(x) = 4x$ is a polynomial, & so is cont. on $(-\infty, \infty)$.

$F(x) = \frac{1}{2}x^3 e^{x^2}$ is also cont. on $(-\infty, \infty)$, since x^3 & e^{x^2} are cont.

$\therefore$ By the Theorem, $\exists!$ solution to the IVP on $(-\infty, \infty)$.

i.e.) A solution exists on $(-\infty, \infty)$ & this solution is unique, so there can't exist more than one solution on any interval, since unique on $(-\infty, \infty)$.

[2pts] 2. Consider the 1^{st}-order DE $(y')^2 + 8 = 0$. Does this eq'n possess any real solutions? i.e. Can there exist a real-valued function $y = \phi(x)$ which satisfies this DE on some interval? Explain.

$(y')^2 + 8 = 0 \Rightarrow \underbrace{(y')^2}_{\geq 0} = \underbrace{-8}_{<0}$. Suppose $y = \phi(x)$ is a real solution.

Then $y = \phi(x)$ is a real-valued function $\Rightarrow$ $y(x) \in \mathbb{R} \ \forall x$.

This means that $y'(x)$ is also a real-valued function $\Rightarrow (y'(x))^2 \geq 0 \ \forall x$.

But then this contradicts $(y')^2 = -8 < 0$.

$\therefore$ No real solution can exist.

[4pts] 3. Suppose you are given a 1^{st}-order DE $y' = f(x,y)$, which satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R. Could 2 different solution curves in its 1-parameter family of solutions intersect at a point in R? Why or why not?

Recall: Theorem 1.2.1: Let $R = [a,b] \times [c,d]$ contain (x_0, y_0) in its interior. If $f(x,y)$ & $\frac{\partial f}{\partial y}$ are cont. on $R \Rightarrow \exists$ some interval $I_0: (x_0 - h, x_0 + h), h > 0$, contained in $[a,b]$, & a unique function $y(x)$, defined in I_0, that is a solution to the IVP $y' = f(x,y)$, $y(x_0) = y_0$.

Theorem 1.2.1 satisfied for a region $R \Rightarrow$ for all points (x_0,y_0) in R there is an interval I_0 containing that point where

the solution to $y' = F(x,y)$, $y(x_0) = y_0$
is unique.

In particular, suppose 2 different solution curves
$G(x,y,c)$ & $G(x,y,\tilde{c})$ intersect at (x_0, y_0).

Restricting these to I_0, we would have two
distinct solution curves going through (x_0, y_0) on
I_0, which is a contradiction. ☺

$\therefore$ $c = \tilde{c}$. i.e., The 2 curves must be the same.

$\therefore$ Two different solution curves can't intersect
at a point (x_0, y_0) in R.

- **First Three Classes:** We focussed on the theory of *first order* DE's. In particular, we learned some techniques for solving special types of first order DE's (separable, linear, exact, substitution methods) and also analyzed first-order DE's geometrically (direction fields, phase portraits).

- **First Three Classes:**We focussed on the theory of *first order* DE's. In particular, we learned some techniques for solving special types of first order DE's (separable, linear, exact, substitution methods) and also analyzed first-order DE's geometrically (direction fields, phase portraits).

- **This Week:** We'll examine *higher-order* **linear** DE's. We'll discuss general theory and techniques for finding a solution.

Things to Review From First-Year Linear Algebra:

- Vector Spaces

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- Vector Spaces
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- Systems of Homogenous Equations (finding a basis for solution space)
- Determinants (at least 2x2 and 3x3 determinants)