Math 2C03 - Class #4

Remark: In 2.2, 2.3, 2.4, 2.5 we focused on solving special types of 1^{st}-order DE's. In the next 2 classes (Ch. 4) we'll learn how to solve nth-order linear DE's.

4.1: Preliminary Theory - Linear Eq'n's:

Def'n: • An nth-order linear IVP has the form

$$(*)\quad a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x), \text{ subject to}$$
$$y(x_0) = y_0,\ y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}.$$

• A solution to this IVP is a function y, C^n on an interval I containing x_0 which satisfies the n initial conditions & the DE.

Theorem 4.1.1 [Existence & Uniqueness]: If $a_n(x), a_{n-1}(x), \dots, a_0(x)$ & $g(x)$ are continuous on an interval $I \ni x_0$ & $a_n(x) \neq 0\ \forall x \in I$, then a solution $y(x)$ of $(*)$ exists on I & is unique.

In what follows, we assume these assumptions are satisfied.

e.g.: Consider $y'' + (\tan x)y = e^x$, $y(0) = 1$, $y'(0) = 0$.

Here $n = 2$. $a_2(x) = 1 \neq 0\ \forall x$. $a_0(x) = -e^x$ continuous everywhere. $a_1(x) = \tan x$ cont. on $(-\frac{\pi}{2}, \frac{\pi}{2})$. ∴ a unique solution exists on $(-\frac{\pi}{2}, \frac{\pi}{2})$. [Also here $x_0 = 0$ & $0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$].

e.g.: Consider $x^2y'' - 2y = 0$, $y(0) = 0$, $y'(0) = 0$.

Here $y = cx^2$ is a solution. [Indeed, $y(0) = 0$, $y'(0) = 2cx|_{x=0} = 0$. $x^2y'' - 2y$

$= x^2(2c) - 2(cx^2) = 0$ ✓]. So, there's infinitely many solutions on $(-\infty, \infty)$. This does not contradict the Theorem, b/c here $n=2$ & $a_2(x) = x^2 = 0$ at $x=0$, & $x_0 = 0$ here.

Def'n: • An nth-order linear DE is called homogeneous if it has the form
$$a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = 0.$$
• An eq'n of the form $a_n(x)y^{(n)} + \dots + a_0(x)y = g(x)$ for $g(x) \neq 0$ is called nonhomogeneous.

Note: The word homogeneous in this context does not mean the same thing as the homog. DE's in 2.5.

Notation: When we take the derivative $\frac{d}{dx}$ of a function $y(x)$, we transform y into a new function y'. Similarly, $\frac{d^2}{dx^2}$ transforms y into a new function y''. Therefore, it's sometimes convenient to think about the derivative as a function D, which maps y to a new function y'.

e.g.: If $y = x^2$, then $D(y) = 2x$ & $D^2(y) = \frac{d^2y}{dx^2} = 2$.

We call D a differential operator.

Similarly, an nth order differential operator $L := a_n(x)D^n + a_{n-1}(x)D^{n-1} + \dots + a_1(x)D + a_0(x)$ is a function, where

(*) $$\left\{ L(y) = a_n(x)\underbrace{D^n(y)}_{\frac{d^ny}{dx^n}} + \dots + a_1(x)\underbrace{D(y)}_{\frac{dy}{dx}} + a_0(x)y \right.$$

So $L(y) = 0$ & $L(y) = g(x)$ is a compact way to write a DE.

* The next few Theorems will allow us to conclude that the space of solutions of a homog. linear DE is an n-dim'l vector space. *

e.g.] If $L = xD^2 + 2D + x^2$, then $L(y) = 0$ is the DE $xy'' + 2y' + x^2 = 0$.

Recall: A map T is called linear if:
[1] $T(x+y) = T(x) + T(y)$
[2] $T(cx) = cT(x)$.

Exercise: L is a linear operator.

By the properties of differentiation this is clear: $D(cF(x)) = cD(F(x))$ & $D(F(x) + g(x)) = D(F(x)) + D(g(x))$.

Theorem 4.1.2 [Superposition Principle - Homog. Eq'n's]:

Let $y_1, \dots, y_K$ be solutions of a homog. n^{th}-order DE $a_n(x)y^{(n)} + \dots + a_0(x) = 0$ on an interval I. Then the linear combination $y = c_1y_1(x) + \dots + c_Ky_K(x)$ is also a solution on I, where c_i constants.

Cor.: The set of solutions of a homog. n^{th}-order DE form a vector space.

[i.e.] The set satisfies all 10 vector space axioms.]

Proof: Consider L as defined in ⊛. $L(y) = L(c_1y_1 + \dots + c_Ky_K) \underset{L \text{ linear}}{=} c_1L(y_1) + \dots + c_KL(y_K)$

y_i solutions, so $L(y_i) = 0$ $\forall i$

$= c_1(0) + \dots + c_K(0) = 0 \Rightarrow y$ is a solution. ■

Recall: Given a vector space V, a set of vectors $v_1, \dots, v_n$ are called <u>linearly independent</u> if

$$c_1 v_1 + \dots + c_n v_n = \vec{0} \iff c_1 = \dots = c_n = 0.$$

In this context, our vector space V is the set of solutions of the homog. eq'n $L(y) = 0$.

Def'n: A set of functions $f_1(x), \dots, f_n(x)$ are <u>linearly independent</u> on an interval I if

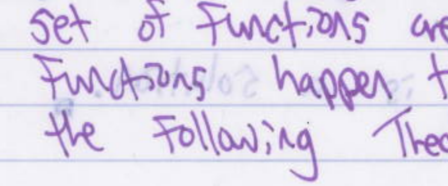
$$c_1 f_1(x) + \dots + c_n f_n(x) = 0 \quad \forall x \in I \iff c_1 = \dots = c_n = 0.$$

A set of functions is <u>linearly dependent</u> if it fails to be linearly independent (i.e. $\exists$ constants c_i <u>not</u> all zero s.t. $c_1 f_1(x) + \dots + c_n f_n(x) = 0 \quad \forall x \in I$).

e.g. • e^{x+2} & e^{x-3} are linearly dependent on $(-\infty, \infty)$. Consider $c_1 = -e^{-5}$, $c_2 = 1$:

$$c_1 e^{x+2} + c_2 e^{x-3} = -e^{-5} e^{x+2} + e^{x-3} = -e^{x-3} + e^{x-3} = 0.$$

• x & $|x|$ are linearly independent on $(-\infty, \infty)$. Indeed, if we consider their graphs, they're clearly not a constant multiple of each other:



* It's not always easy to tell whether or not a set of functions are linearly independent. If our functions happen to be solutions of $L(y) = 0$, then the following Theorem gives us an easy check. *

Def^n: If $f_1, \dots, f_n$ possess at least $n-1$ derivatives, then the determinant

$$W(f_1, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & \dots & & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

is the **Wronskian** of the functions.

Theorem 4.1.3 [Criterion for Linear Independent Solutions]:

A set of solutions $\{y_1, \dots, y_n\}$ of the homog. linear eq^n $L(y)=0$ on I are linearly independent on I $\iff$ $W(y_1, \dots, y_n) \neq 0$ $\forall x \in I$.

Theorem: A set of linearly independent solutions $\{y_1, \dots, y_n\}$ to $L(y)=0$ form a basis for the solution space.

i.e., let V be the vector space of solutions to $L(y)=0$. Then $\dim V = n$. [i.e., $\{y_1, \dots, y_n\}$ span V].

[i.e., any $y \in V$ has the form $y = c_1 y_1 + \dots + c_n y_n$].

Proof: [The existence of such a set $y_1, \dots, y_n$ is stated without proof in Theorem 4.1.4 & a partial proof for $\operatorname{span}\{y_1, \dots, y_n\} = V$ is given in Theorem 4.1.5].

Def^n: A basis $\{y_1, \dots, y_n\}$ for $L(y)=0$ is called a **Fundamental set of solutions**.

Def^n: Given a fundamental set of solutions to $L(y)=0$ on I, the general solution of $L(y)=0$ on I is

$y = c_1 y_1(x) + \dots + c_n y_n(x)$, where c_i arbitrary constants.

[i.e., any solution has this form for some constants c_i].

Def^n: A solution y_p to $\overbrace{L(y) = g(x)}^{\text{non-homog. DE}}$ free of arbitrary parameters is called a particular solution.

$L(y) = g(x)$ means $a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = g(x)$.

Theorem 4.1.6 [General Solution of Nonhomog. Eq'n]:

let y_p be a particular solution to the nonhomog. linear DE $L(y) = g(x)$ on I & $\{y_1, \dots, y_n\}$ the fundamental set of solutions of $L(y) = 0$ on I. Then the general solution of $L(y) = g(x)$ on I is:

$$\underbrace{y = c_1 y_1 + \dots + c_n y_n + y_p.}$$

[i.e. all possible solutions have this form for some c_i's]

e.g. Verify $\{e^t, te^t\}$ is a fundamental solution set of $y'' - 2y' + y = 0$ for $(-\infty, \infty)$. Find the general solution.

First we need to make sure they satisfy the DE:

e^t: $e^t - 2e^t + e^t = 0.$ ✓

$y = te^t$
$y' = e^t + te^t$
$y'' = 2e^t + te^t$

te^t: $2e^t + te^t - 2e^t - 2te^t + te^t = 0.$ ✓

Now we need to make sure $W(e^t, te^t) \neq 0$:

$$W(e^t, te^t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t} \neq 0 \text{ on } (-\infty, \infty).$$

$\therefore$ e^t & te^t are linearly independent & form a fundamental set.

$\therefore$ $y = c_1 e^t + c_2 te^t$ is the general solution on $(-\infty, \infty)$.

e.g. Find the general solution to the nonhomog. eq'n $y'' - 2y' + y = 8$.

By inspection, $y = 8$ is a particular solution. Since $\{e^t, te^t\}$ is fundamental set of $y'' - 2y' + y = 0$, the general solution is $y = c_1e^t + c_2te^t + 8$.

$\boxed{4.2}$ Reduction of order:

In this section, we will outline a method for solving 2^{nd}-order linear homogeneous DE's $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$.

Idea: If we already know one solution to this DE, $y_1(x)$, then our second-order DE can be reduced to a linear eq^n by making an appropriate substitution involving $y_1(x)$.

* This substitution is outlined explicitly on pg. 130, & the derivation of the formula below is also given on this pg.. It's not hard, but for the sake of time we will take the following formula as given. *

Method of Solution:

$\boxed{1}$ Write the 2^{nd}-order homog. DE in the form $y'' + P(x)y' + Q(x)y = 0$.

$\boxed{2}$ Find one solution to this DE, $y_1(x)$. [In this section this known solution $y_1(x)$ will be given to us].

$\boxed{3}$ The second solution is given by:

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx.$$

$\boxed{\text{e.g. 7}}$ Find the general solution of $y'' \overbrace{-4}^{P(x)}y' + \overbrace{3}^{Q(x)}y = 0$, where it is known that $y_1(x) = e^x$ is a solution.

$$y_2 = e^x\int \frac{e^{-\int -4\,dx}}{e^{2x}}\,dx = e^x\int \frac{e^{4x}}{e^{2x}}\,dx = e^x\int e^{2x}\,dx$$

$= e^x [\tfrac{1}{2} e^{2x} + c]$.

So, a second solution would be $\boxed{y_2(x) = e^{3x}}$.
[Any c would be fine, + any constant multiple].

$\therefore y = c_1 e^x + c_2 e^{3x}$ is a general solution.

$\boxed{\text{4.3: Homogeneous Linear Eq'n's w/ Constant Coefficients}}$

In this section we'll learn how to solve homogeneous linear eq'n's w/ constant coefficients i.e., eq'n of the form $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$, where $a_i \in \mathbb{R}$.

$\boxed{\text{Idea:}}$ Notice that in the $n=1$ case $a_1 y' + a_0 y = 0$, $\Leftrightarrow y' = -\frac{a_0}{a_1} y$
we're looking for a solution y whose derivative is a constant multiple of itself, so $y = e^{mx}$ is a natural candidate: $a_1 m e^{mx} + a_0 e^{mx} = 0 \Leftrightarrow a_1 m + a_0 = 0$.
So we just need to solve a polynomial eq'n!

In the general case, we do a similar thing: $y = e^{mx}$ is a natural candidate for a solution,
so to find appropriate m, we need to solve the polynomial eq'n $\underbrace{a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 = 0}_{\text{auxiliary eq'n}}$.

If we get n distinct roots $m_1, \dots, m_n$, then $e^{m_i x}$ is a solution for each i, so $y = c_1 e^{m_1 x} + \dots + c_n e^{m_n x}$ would be the general solution of our DE.

If we have repeated roots or complex roots, then we need to do a bit more work. Let's examine this in the $n = 2$ case.

$\boxed{n=2:}$ $a_2y'' + a_1y' + a_0y = 0.$

Try: $y = e^{mx}$: $e^{mx}[a_2m^2 + a_1m + a_0] = 0 \iff a_2m^2 + a_1m + a_0 = 0$

$\iff m = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$. We'll either have 2 real, 2 complex, or a repeated real root.

- $\boxed{\text{Case 1:}}$ 2 distinct real roots m_1 & m_2: $[a_1^2 - 4a_2a_0 > 0]$

e^{m_1x} & e^{m_2x} solutions. $\begin{vmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{vmatrix} = (m_2 - m_1)e^{(m_1+m_2)x} \neq 0$ $\because$ since $m_1 \neq m_2$.

So, $y = c_1e^{m_1x} + c_2e^{m_2x}$ general solution.

- $\boxed{\text{Case 2:}}$ Repeated Real Roots m_1: $[a_1^2 - 4a_2a_0 = 0]$

$y_1 = e^{m_1x}$ solution. By Reduction of Order: $[y'' + \underbrace{\tfrac{a_1}{a_2}}_{P(x)}y' + \underbrace{\tfrac{a_0}{a_2}}_{Q(x)}y = 0]$

$y_2 = e^{m_1x}\int \frac{e^{-\frac{a_1}{a_2}x}}{e^{2m_1x}}\,dx = e^{m_1x}\int \frac{e^{2m_1x}}{e^{2m_1x}}\,dx$ $\quad m_1 = -\frac{a_1}{2a_2}$

$= e^{m_1x}[x + c]$. $\therefore y_2 = xe^{m_1x}$ also a solution.

General solution is $y = c_1e^{m_1x} + c_2xe^{m_1x}$.

- $\boxed{\text{Case 3}}$ 2 complex roots m_1 & m_2: $[a_1^2 - 4a_2a_0 < 0]$

$m_1 = \alpha + \beta i$
$m_2 = \alpha - \beta i$

General solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$, where $m_1, m_2 \in \mathbb{C}$.

We can write this general solution in terms of real solutions:

$\boxed{\text{Recall:}}$ $e^{i\theta} = \cos\theta + i\sin\theta$. $\rightsquigarrow$ Euler's Formula

Choose $y_1 = \frac{1}{2}e^{m_1x} + \frac{1}{2}e^{m_2x} \overset{*}{=} \frac{1}{2}e^{\alpha x}[e^{\beta ix} + e^{-\beta ix}]$

m_1 & m_2 complex conjugate roots $\Rightarrow m_1 = \alpha + \beta i$ & $m_2 = \alpha - \beta i$

$= \frac{1}{2} e^{\alpha x} [\cos\beta x + i\sin\beta x + \cos\beta x - i\sin\beta x] = e^{\alpha x} \cos\beta x$ [real solution].

Similarly, choose $y_2 = -\frac{i}{2} e^{m_1 x} + \frac{i}{2} e^{m_2 x} = -\frac{i}{2} e^{\alpha x} [i\sin\beta x + i\sin\beta x]$
$= e^{\alpha x} \sin(\beta x)$ [real solution].

$e^{\alpha x} \cos(\beta x)$ & $e^{\alpha x} \sin(\beta x)$ are linearly independent (check),
so we can write the general solution as

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x).$$

General Case: In a similar way, for the n^{th}-order eq'n, if all roots are distinct the general solution is $y = c_1 e^{m_1 x} + \dots + c_n e^{m_n x}$.

If m_1 is a root of multiplicity k : the linearly independent solutions are $e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \dots, x^{k-1} e^{m_1 x}$
i.e. The general solution contains the linear combination
$c_1 e^{m_1 x} + c_2 x e^{m_1 x} + \dots + c_k x^{k-1} e^{m_1 x}$.

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e.g.: Find the general solution of the following:

(a) $y'' - 36y = 0$

$m^2 - 36 = 0 \Leftrightarrow (m+6)(m-6) = 0 \Leftrightarrow m = \pm 6$.

$y = c_1 e^{6x} + c_2 e^{-6x}$.

(b) $y'' - 10y' + 25y = 0$

$m^2 - 10m + 25 = 0 \Leftrightarrow (m-5)^2 = 0 \Leftrightarrow m = 5$.

$y = c_1 e^{5x} + c_2 x e^{5x}$.

(c) $2y'' + 2y' + y = 0.$

$2m^2 + 2m + 1 = 0 \iff m = \frac{-2 \pm \sqrt{4-8}}{4} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-1 \pm i}{2}.$

$y = c_1 e^{-x/2}\cos(\tfrac{1}{2}x) + c_2 e^{-x/2}\sin(x/2).$ $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$

(d) $y^{(4)} - 2y'' + y = 0.$

$m^4 - 2m^2 + 1 = 0 \iff (m^2-1)^2 = 0 \iff m^2 = 1 \iff m = \pm 1.$

$y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{-x} + c_4 x e^{-x}.$

(e) $y^{(4)} - 7y'' - 18y = 0.$

$3 \quad 18 \quad \frac{4}{72}$

$m^4 - 7m^2 - 18 = 0 \iff (m^2-9)(m^2+2) = 0 \iff m = \pm 3$ or $m = \pm\sqrt{2}\, i.$

$y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos(\sqrt{2}x) + c_4 \sin(\sqrt{2}x).$

(f) $y''' + 3y'' - 4y' - 12y = 0.$

$m^3 + 3m^2 - 4m - 12 = 0.$ By inspection, $m = 2$ root.

$$\begin{array}{r} m^2 + 5m + 6 \\ (m-2)\overline{\big) m^3 + 3m^2 - 4m - 12} \\ \underline{m^3 - 2m^2} \\ 5m^2 - 4m - 12 \\ \underline{5m^2 - 10m} \\ 6m - 12 \\ \underline{6m - 12} \\ 0 \end{array}$$

$m^3 + 3m^2 - 4m - 12 = (m-2)(m^2+5m+6)$
$= (m-2)(m+2)(m+3).$

$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{-3x}.$