Math 2C03 - Class #3

2.5: Solutions By Substitution

Last class: We discussed methods for solving 3 special types of 1^{st}-order DE's: separable (2.2), linear (2.3), & exact (2.4).

If we encounter a 1^{st}-order DE that doesn't fall into one of these 3 categories, sometimes we can make a substitution to transform the DE into one of our 3 types.

Def^n: A function $F:\mathbb{R}^2 \to \mathbb{R}$ is called homogeneous if $F(tx, ty) = t^\alpha F(x,y)$ for some $\alpha \in \mathbb{R}$.

e.g. → • $F(x,y) = x^2+y^2$ is homog. of degree 2, since $F(tx,ty) = t^2x^2 + t^2y^2 = t^2(x^2+y^2) = t^2F(x,y)$.

• $F(x,y) = x^2+y$ is not homog., since $F(tx,ty) = t^2x^2+ty = t(tx^2+y)$ ... no way to write in form $F(tx,ty) = t^\alpha F(x,y)$.

Def^n: A 1^{st}-order DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is homogeneous if M & N are homog. functions of the same degree.

Solving 1^{st}-order Homog. DE's (via substitution):

$\boxed{1}$ Make the substitution $\underbrace{y = ux,\quad dy = x\,du + u\,dx}_{\text{try when } N \text{ simpler than } M}$ or $\underbrace{x = vy,\quad dx = y\,dv + v\,dy}_{\text{try when } M \text{ simpler than } N}$.

$\boxed{2}$ Rearrange to see that your eq^n into a separable eq^n.

[3] Solve the separable eq'n.

[4] Substitute back to put your answer in terms of x & y.

e.g.) $\frac{dy}{dx} = \frac{x^2-y^2}{3xy}$.

$$\underbrace{3xy}_{N}\,dy + \underbrace{(-x^2+y^2)}_{M}\,dx = 0.$$

M & N are both homog. of deg. 2, since

$M(tx,ty) = -t^2x^2 + t^2y^2 = t^2(-x^2+y^2) = t^2 M(x,y)$ &

$N(tx,ty) = 3tx\,ty = t^2\,3xy = t^2 N(x,y)$.

$$\left.\begin{aligned} y &= ux \\ dy &= x\,du + u\,dx. \end{aligned}\right\} \Rightarrow (-x^2+u^2x^2)\,dx + 3x^2u\,(x\,du + u\,dx) = 0$$

$$\Rightarrow (-x^2 + 4u^2x^2)\,dx + 3x^3u\,du = 0$$

$$\Rightarrow \tfrac{1}{x}(-1+4u^2)\,dx + 3u\,du = 0$$

$$\Rightarrow \int \tfrac{1}{x}\,dx = \int \frac{-3u}{4u^2-1}\,du$$

$w = 4u^2-1$

$dw = 8u\,du$

$\tfrac{1}{8}dw = u\,du$

$$\Rightarrow \ln x = -\tfrac{3}{8}\int \tfrac{1}{w}\,dw$$

$$\Rightarrow \ln x = -\tfrac{3}{8}\ln(4u^2-1) + c$$

$$\Rightarrow \ln x = \ln\left((4u^2-1)^{-3/8}\right) + c$$

$$\Rightarrow x = c(4u^2-1)^{-3/8}$$

$$\Rightarrow \boxed{x = c\left(4\tfrac{y^2}{x^2}-1\right)^{-3/8}}$$

implicit solution.

$$\Rightarrow x^8\left(4\tfrac{y^2}{x^2}-1\right)^3 = c$$

$$\Rightarrow x^8\left(\tfrac{1}{x^2}(4y^2-x^2)\right)^3 = c$$

$$\Rightarrow x^2(4y^2-x^2)^3 = c.$$

Def^{n}: Bernoulli's eq^{n} is a DE of the form

$$\frac{dy}{dx} + P(x)y = F(x)y^n, \text{ where } n \in \mathbb{R}.$$

[if $n=0$ or $n=1$, then this eq^{n} is linear].

Solving Bernoulli's Eq^{n}:

1. Make the substitution $u = y^{1-n}$ $\therefore$ $du = (1-n)y^{-n}\,dy$.
2. Solve this linear eq^{n}. [$y^{-n}dy + P(x)y^{1-n}dx = F(x)dx$; $\frac{du}{dx} + (1-n)P(x)u = (1-n)F(x)$ linear].
3. Write equation in terms of x & y.

e.g.7 Solve $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$.

Here $n=3$, so let $u = y^{1-3} = y^{-2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$\frac{1}{2}\frac{du}{dx} = y^{-3}\frac{dy}{dx}$$

$$\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$$

$$y^{-3}\frac{dy}{dx} - 5y^{-2} = -\frac{5}{2}x$$

$$-\frac{1}{2}\frac{du}{dx} - 5u = -\frac{5}{2}x$$

$$\frac{du}{dx} + \underbrace{10}_{P(x)}u = \underbrace{5x}_{F(x)}$$

$$\int P(x)dx = \int 10\,dx = 10x.$$

$w = x$, $dw = dx$, $v = \frac{1}{10}e^{10x}$, $dv = e^{10x}$

$$u = e^{-10x}\left[\int e^{10x}(5x)\,dx\right]$$

$$= 5e^{-10x}\left[\frac{1}{10}xe^{10x} - \frac{1}{10}\int e^{10x}dx\right]$$

$$= \frac{e^{-10x}}{2}\left[xe^{10x} - \frac{1}{10}e^{10x} + c\right]$$

$$= \frac{x}{2} - \frac{1}{20} + ce^{-10x}.$$

$$\therefore \boxed{y^{-2} = \frac{x}{2} - \frac{1}{20} + ce^{-10x}}$$

implicit solution

[An example of a DE not in this form would be $\frac{dy}{dx} = (x+y)^2 + x^2$.]

Reduction to Separation of Variables:

A DE of the form $\frac{dy}{dx} = F(Ax+By+c)$, for $A, B, C \in \mathbb{R}$, $\boxed{B \neq 0}$

can be reduced to a separable eq'n via the substitution $u = Ax + By + c$.

e.g. $\frac{dy}{dx} = (-2x+y)^2 - 7$ has this form.

The function $F(u) = u^2 - 7$. $u = -2x + y$. So, $A = -2$, $B = 1$, $C = 0$.

[This example is worked out on pg. 73].

e.g. $\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}$.

Here the function is $F(u) = -u - 1 + (u+2)^{-1}$. $u = x - y$

$[A = 1, B = -1, C = 0]$.

$\frac{du}{dx} = 1 - \frac{dy}{dx}$.

$\frac{dy}{dx} = 1 - \frac{du}{dx}$.

$1 - \frac{du}{dx} = -u - 1 + (u+2)^{-1}$

$\frac{du}{dx} = u + 2 - \frac{1}{u+2}$

$w = (u+2)^2 - 1$

$dw = 2(u+2)\,du$

$\frac{1}{2}dw = (u+2)\,du$

$\frac{du}{dx} = \frac{(u+2)^2 - 1}{u+2}$

$\int \frac{(u+2)}{(u+2)^2 - 1}\,du = \int dx$

$\frac{1}{2}\int \frac{1}{w}\,dw = x + c$

$\ln((u+2)^2 - 1) = 2x + c$

$(u+2)^2 = ce^{2x} + 1$

$\boxed{(x-y+2)^2 = ce^{2x} + 1}$.

implicit solution

Recap: Homog. DE's $\overset{u=\frac{y}{x} \text{ or } v=\frac{x}{y}}{\rightsquigarrow}$ Separable

Bernoulli's Eq'n $\overset{u=y^{1-n}}{\rightsquigarrow}$ linear

$\frac{dy}{dx} = F(Ax+By+C)$ $\overset{u=Ax+By+C}{\rightsquigarrow}$ Separable

2.1: Solution Curves Without a Solution

- The past few classes we've been focusing on analytically solving 1st-order DE's. However, sometimes there is no way to symbolically write down what the solution is. In fact, in most real-world applications, this is the case!

Therefore, it's useful to analyze the behavior of solutions of DE's qualitatively (i.e. Existence & Uniqueness Theorems) & geometrically (direction fields, phase portraits).

Def'n: A direction field of the DE $\frac{dy}{dx} = F(x,y)$ is the vector field $F(x_0,y_0) = (1, \frac{dy}{dx}(x_0,y_0))$.
i.e. Each pt (x_0,y_0) in the xy-plane corresponds to a vector $(1, \frac{dy}{dx}(x_0,y_0))$.

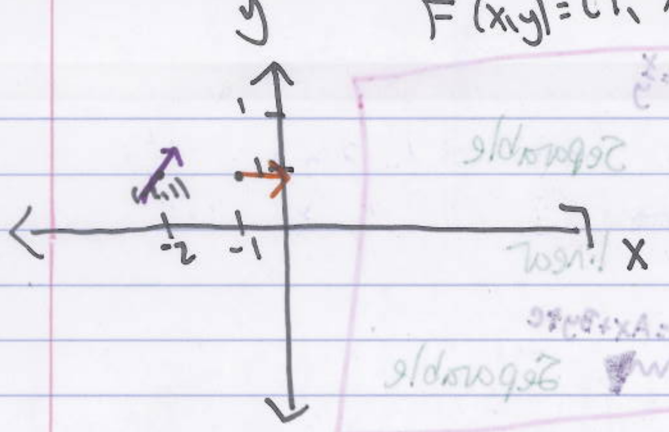
Here $\frac{dy}{dx}(x_0,y_0) = F(x_0,y_0)$ is the slope of the tangent line at $(x_0, y(x_0))$ on the solution curve.

e.g. $\frac{dy}{dx} = x^2 - y$.

The graph of a solution to this DE that passes through the point $(-2,1)$ must have a slope $(-2)^2 - 1 = 3$.
i.e. $F(-2,1) = (1,3)$. $F(-1,1) = (1, (-1)^2 - 1) = (1,0)$.

To graph these, identify the point $(-2,1)$, & draw the vector $(1,3)$ at that pt.

$F(x,y) = \langle 1, x^2 - y\rangle$.



[Print this VF & give as handout].

- A single solution curve that passes through a direction field must follow the flow pattern of the field. i.e. at each point the curve must be tangent to the slope vector at that point.

Def^n: An <u>autonomous</u> DE is a DE in which the independent variable does not appear explicitly. i.e. If x is independent & y dependent, then an autonomous 1st-order DE has the form $\frac{dy}{dx} = F(y)$.

e.g. $\frac{dy}{dx} = \underbrace{2y+7}_{F(y)}$ is autonomous, but

$\frac{dy}{dx} = xy$ is <u>not</u>.

Notice: Autonomous DE's are separable!

* We're going to look at autonomous DE's b/c
1. Autonomous DE's appear in lots of real-world applications.
2. They're relatively easy to analyze geometrically. *

Def^n: Given $\frac{dy}{dx} = F(y)$, the zeros of F (i.e. $\{c \in \mathbb{R} \mid F(c) = 0\}$) are called <u>critical points</u> (<u>a.k.a.</u> <u>equilibrium points</u>).

<u>Notice</u>: If c is a critical point, then $y(x) = c$ is a solution to the DE $y' = F(y)$.

Recall: Given a function $y(x)$, the sign of the derivative $\frac{dy}{dx}$ determines where $y(x)$ is increasing / decreasing.

Given an autonomous DE $y' = F(y)$, we keep track of where the solutions of this DE are increasing / decreasing in <u>phase portrait</u>:

Assuming F & F' cont. on some interval I.

e.g.⟹ Consider the autonomous DE $\frac{dy}{dx} = \underbrace{y^2 - y^3}_{F(y)}$.

$y^2 - y^3 = 0 \iff y^2(1-y) = 0 \iff y = 0$ or $y = 1$.

∴ The critical pts are 0 & 1.

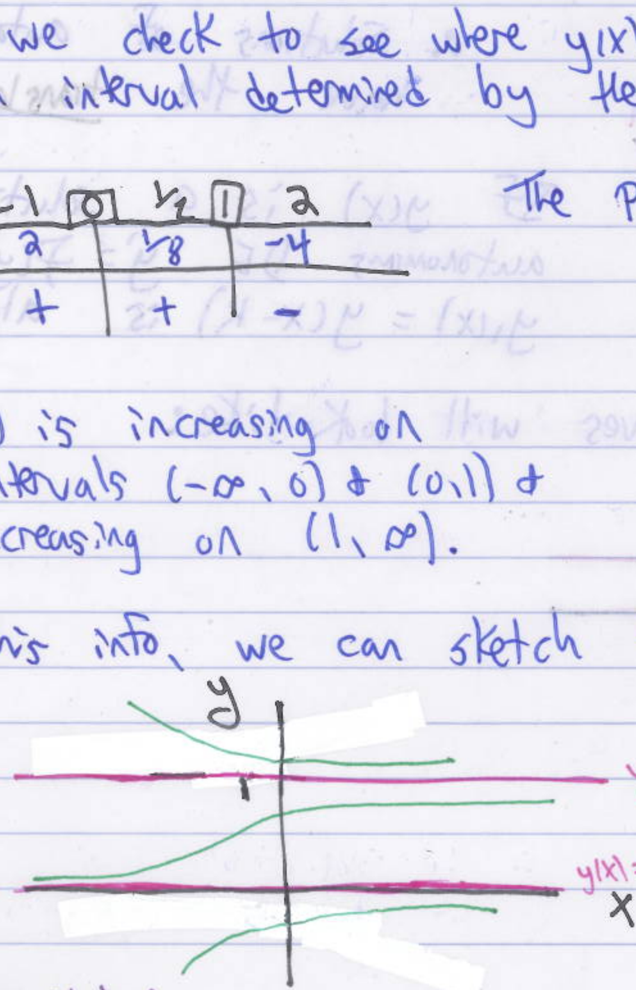
Now we check to see where $y(x)$ is increasing / decreasing on each interval determined by these critical pts:

y	-1	$\boxed{0}$	$\frac{1}{2}$	$\boxed{1}$	2
$y' = F(y)$	2		$\frac{1}{8}$		-4
sign	$+$		$+$		$-$

The phase portrait is:

i.e.⟹ $y(x)$ is increasing on the intervals $(-\infty, 0)$ & $(0, 1)$ & is decreasing on $(1, \infty)$.

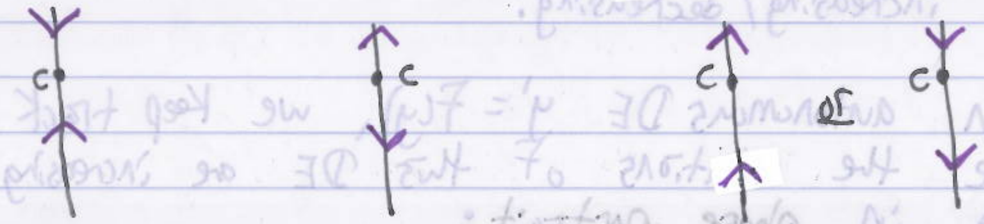
With this info, we can sketch the solution curves:



∴ A nonconstant solution must lie entirely on one side of an equilibrium solution.

A nonconstant solution can't cross an equilibrium solution, b/c if it did, then the pt it crosses at has 2 unique IVP solutions. ↯

Def'n: Critical points c can either be <u>attractors</u> (a.k.a. <u>stable</u>), <u>repellers</u> (a.k.a. <u>unstable</u>), or <u>semi-stable</u>:



attractor (stable) repeller (unstable) semi-stable

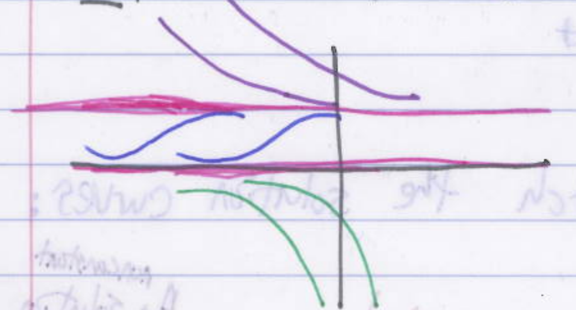
Notice: In the direction field corr. to an autonomous DE, the slopes of the vectors on a horizontal line will all be the same (since $F(y)$ does not depend on x).

e.g.

$\therefore$ Solutions of autonomous DE's have the <u>translation property</u>:

If $y(x)$ is a solution of an autonomous DE $y' = F(y)$, then $y_1(x) = y(x-K)$ is also a solution, $K \in \mathbb{R}$.

i.e. Solution curves will look like:



Application: Logistic Model [for more info, see 3.2].

Suppose 4 tons of fish are harvested from a fishery per month. A model for the population $P(t)$ of the fishery at month t is given by

$$\frac{dP}{dt} = P(5-P) - 4, \quad P(0) = P_0,$$

where P_0 is the population of fish in tons at month zero.

[a] Use phase portraits to sketch solution curves.

[b] Determine the long-term behaviour of the population.

[c] Solve the IVP.

[d] Will the fishery population become extinct in finite time?

---

[a] Critical points are given when $P(5-P)-4=0$

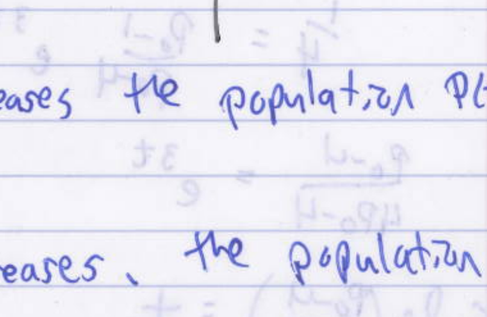
$\Leftrightarrow 5P - P^2 - 4 = 0 \Leftrightarrow P^2 - 5P + 4 = 0 \Leftrightarrow (P-1)(P-4) = 0$

$\Leftrightarrow P = 1$ or $P = 4$.

0 1 2 4 5

$- \mid + \mid -$

$\therefore$ 1 repellor & 4 attractor.



[b] If $0 < P_0 < 1$, then as t increases the population $P(t)$ tends to 0.

If $1 < P_0 < 4$, then as t increases, the population tends to 4 tons.

If $P_0 > 4$, then the population also tends to 4 tons.

$\boxed{c}$ $\frac{dP}{dt} = -P^2 + 5P - 4$

$\frac{dP}{dt} = -(P-1)(P-4)$

$\int \frac{-1}{(P-1)(P-4)}\, dP = \int dt$

$\int \frac{-1/3}{(P-1)} + \frac{1/3}{(P-4)}\, dP = t + c$

$\frac{1}{3}\ln|P-1| - \frac{1}{3}\ln|P-4| = t + c$

$\ln\left|\frac{P-1}{P-4}\right| = 3t + c$

$\frac{P-1}{P-4} = c\, e^{3t}$

$\frac{-1}{(P-1)(P-4)} = \frac{A}{(P-1)} + \frac{B}{(P-4)}$

$\Rightarrow -1 = A(P-4) + B(P-1)$

$\Rightarrow -1 = (A+B)P + (-4A - B)$

$\Rightarrow A + B = 0 \;\&\; -4A - B = -1$

$\Rightarrow A = -B \;\&\; 4B - B = 1$

$\Rightarrow 3B = 1$

$\Rightarrow B = 1/3 \;\&\; A = -1/3.$

$P(0) = P_0 \Rightarrow c = \frac{P_0 - 1}{P_0 - 4}.$

$$\therefore \underbrace{\frac{P-1}{P-4} = \frac{P_0-1}{P_0-4}\, e^{3t}}_{\text{implicit solution}}.$$

$\boxed{d}$ From $\boxed{b}$ we know the population tends to 0 for $0 < P_0 < 1$. Take eq'n from $\boxed{c}$ & solve for t when $P = 0$ (i.e. $P = 0$ means population extinct).

$\frac{1}{4} = \frac{P_0-1}{P_0-4}\, e^{3t}$

$\frac{P_0-4}{4P_0-4} = e^{3t}$

$\frac{1}{3}\ln\left(\frac{P_0-4}{4P_0-4}\right) = t.$

$\therefore$ The population will become extinct at $t = \frac{1}{3}\ln\left(\frac{P_0-4}{4(P_0-1)}\right)$ months, if $0 < P_0 < 1$.