Math 2C03 - Class #2

2.2: Seperable Eq^{n}'s

Remark: In sections 2.2 - 2.5 we will discuss strategies for solving special types of 1st-order DE's.

Def^{n}: A seperable eq^{n} is a 1st-order DE of the form $\frac{dy}{dx} = g(x)h(y)$.

e.g.1 • $y' = \underbrace{x^2}_{g(x)} \underbrace{e^y}_{h(y)}$ is seperable.

• $y' = y + \sin x$ is not seperable.

Method of Solution:

$$y' = g(x)h(y)$$

$$\frac{1}{h(y)} y' = g(x)$$

$$\underbrace{\int \frac{1}{h(y)}\, dy = \int g(x)\, dx}.$$

this gives us the solution.

Why?: [see next page].

e.g.1 Solve $y' = \underbrace{x^2}_{g(x)} \underbrace{e^y}_{h(y)}$.

$$e^{-y} y' = x^2$$

$$\int e^{-y}\, dy = \int x^2\, dx$$

$$-e^{-y} = \tfrac{1}{3} x^3 + c$$

$$e^{-y} = -\tfrac{1}{3} x^3 + c$$

$$\ln(e^{-y}) = \ln\left(-\tfrac{1}{3} x^3 + c\right)$$

$$-y = \ln\left(-\tfrac{x^3}{3} + c\right)$$

$$y = -\ln\left(-\tfrac{x^3}{3} + c\right)$$

$$\boxed{y = \ln\left(\frac{1}{-\frac{x^3}{3} + c}\right)}.$$

To see where defined need to check where c!

$$y' = \frac{-1}{-x^3/3 + c} \cdot (-x^2) = \frac{x^2}{-x^3/3 + c}.$$

We need $-\frac{x^3}{3}+c>0$ in order for y to be defined, & $-\frac{x^3}{3}+c\neq 0$ for y' to be continuous.

$\therefore y=-\ln(c-\frac{x^3}{3})$ is a solution on any interval where $c-\frac{x^3}{3}>0$.

Why do we solve in this way?

Let $\frac{1}{h(y)}=: p(y)$. We begin with $p(y)\frac{dy}{dx}=g(x)$.

Suppose $y=\phi(x)$ is a solution of $p(y)\,y'=g(x)$.

$$\int p(y)\frac{dy}{dx}\,dx=\int g(x)\,dx$$

Sub. $y=\phi(x)$. $\frac{dy}{dx}=\phi'(x)$ $\Rightarrow$

$$\int p(\phi(x))\underbrace{\phi'(x)\,dx}_{dy}=\int g(x)\,dx$$

$y=\phi(x)$ $\Rightarrow$ $dy=\phi'(x)\,dx$

$$\int p(y)\,dy=\int g(x)\,dx$$

i.e. $H(y)=G(x)+c$, where $H(y)$ is the antider. of $p(y)$ & $G(x)$ anti-der. of $g(x)$.

[by implicit diff.: $\frac{d}{dx}H(y)=\frac{d}{dx}(G(x)+c)$

$p(y)\frac{dy}{dx}=g(x)$, so $H(y)=G(x)+c$ implicit solution].

Singular Solutions: Given a separable DE $y'=g(x)h(y)$, we divide by $h(y)$ & integrate: $\int\frac{1}{h(y)}\,dy=\int g(x)\,dx$. What happens for those constant functions $y=a$, $a\in\mathbb{R}$ that make $h(y)$ vanish (i.e. $h(a)=0$)? If $y=a$ & $h(a)=0$, we have $\frac{dy}{dx}=0$, so $y=a$ satisfies the DE ($0=g(x)(0)$). $\therefore y=a$ is a solution. But using separation of variables may not produce this solution. & therefore $y=a$ may be a singular solution.

$\boxed{\text{e.g.7}}$ $y' + 2xy^2 = 0$. Solve this DE.

$$y' = -2xy^2$$

we notice here that $y=0$ might be a singular solution

$$\int \frac{1}{\boxed{y^2}}\,dy = \int -2x\,dx$$

$$-y^{-1} = -x^2 + c$$

$$1/y = x^2 + c$$

$$y = \frac{1}{x^2+c}.$$

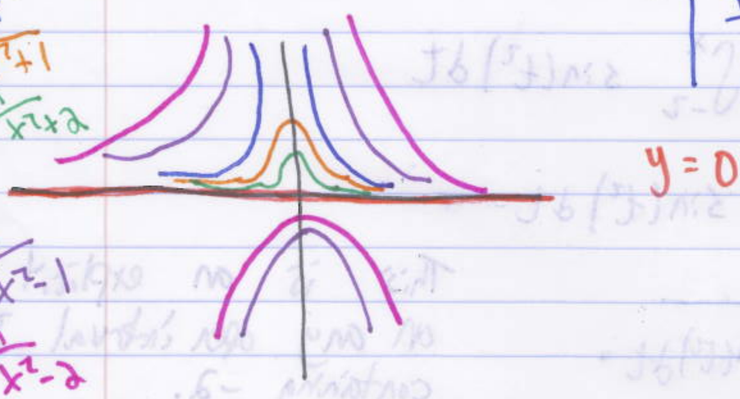
$y=0$ is also a solution, since $y'=0$, so

$0 + 2x(0)^2 = 0.$ ✓

But there's no c we can choose to make $y = \frac{1}{x^2+c}$ the constant zero function.

$\therefore$ $y=0$ is a singular solution.

- $y = \frac{1}{x^2}$
- $y = \frac{1}{x^2+1}$
- $y = \frac{1}{x^2+2}$
- $y = \frac{1}{x^2-1}$
- $y = \frac{1}{x^2-2}$



$\boxed{2.3: \text{Linear Eq}^{n}\text{'s [1}^{st}\text{-order]}}$

$\boxed{\text{Def}^{n}}$: A **1^{st}-order linear DE** is an eq^n of the form $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$.

$\boxed{\text{Solving a Linear } 1^{st}\text{-order DE:}}$

1. Put DE in the form $\frac{dy}{dx} + P(x)y = F(x)$.

2. The 1-par. family of solutions is given by:

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} F(x)\,dx\right]$$

$\boxed{\text{e.g. 7}}$ Solve $xy' - y = 2x\ln x$.

$$y' + \underbrace{\left(-\frac{1}{x}\right)}_{P(x)}y = \underbrace{2\ln x}_{F(x)}$$

$$\int P(x)\,dx = \int -\frac{1}{x}\,dx = -\ln x.$$

$$y = e^{\ln x}\left[\int e^{-\ln x}(2\ln x)\,dx\right]$$

$u = \ln x$
$du = \frac{1}{x}dx$

$$= x\left[\int x^{-1}(2\ln x)\,dx\right]$$

$$= x\int 2u\,du = x\left[u^2 + c\right] = x\left[(\ln x)^2 + c\right].$$

y is defined on $(0,\infty)$. $y' = (\ln x)^2 + c + x\left[\frac{2\ln x}{x}\right]$ cont. on $(0,\infty)$.

$\therefore y = x(\ln x)^2 + cx$ solution on $(0,\infty)$.

Check: $xy' - y = x[(\ln x)^2 + c + 2\ln x] - x(\ln x)^2 - cx$

$= 2x\ln x.$ ✓

Idea Behind Solution: One can verify that

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} F(x)\,dx\right] \ (*)$$ is a solution

of the DE $y' + P(x)y = F(x)$, on an interval I where both $P(x)$ & $F(x)$ are continuous.

Indeed, $y' = \overset{\text{Product rule}}{(-P(x))} e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} F(x)\,dx\right]$

$+ e^{-\int P(x)dx} e^{\int P(x)dx} F(x)$

So y' is C^1 if $P(x)$ & $F(x)$ are cont.

$= -P(x)\underbrace{e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} F(x)\,dx\right]}_{y} + F(x)$

$= -P(x)y + F(x) \implies y' + P(x)y = F(x).$ ✓

On the other hand, suppose y is a solution of $y' + P(x)y = F(x)$ on some interval. We will show such a solution y must have the form $(*)$.

The trick is the following: We begin with $y' + P(x)y = F(x)$. When we multiply both sides of this eq'n by $e^{\int P(x)dx}$ something nice happens:

$$\underbrace{y' e^{\int P(x)dx} + e^{\int P(x)dx} P(x)y}_{= \frac{d}{dx}\left[e^{\int P(x)dx} y\right]} = e^{\int P(x)dx} F(x)$$

$\frac{d}{dx}\left[e^{\int P(x)dx} y\right]$ by the product rule

$\Leftrightarrow \frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} F(x)$

$\Leftrightarrow \int \frac{d}{dx}\left[e^{\int P(x)dx} y\right] dx = \int e^{\int P(x)dx} F(x)\, dx$

$\Leftrightarrow e^{\int P(x)dx} y = \int e^{\int P(x)dx} F(x)\, dx$

$\Leftrightarrow y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} F(x)\, dx\right] = \circledast.$

$\boxed{\bigstar}$ [∴ We showed $\circledast$ is a solution to $y' + P(x)y = F(x)$ & all solutions of $y' + P(x)y = F(x)$ have the form $\circledast$.

Def^n: A general solution on an interval I of a 1^{st}-order DE is a 1-parameter family of solutions s.t. every solution of the DE on I can be obtained from the 1-par. family for an appropriate choice of c.

$\boxed{\text{Theorem}}$ [Existence & Uniques of 1^{st}-order Linear IVP's]:

Consider the IVP $y' + P(x)y = F(x)$, $y(x_0) = y_0$. If $P(x)$ & $F(x)$ are continuous on an interval I containing x_0, then $\exists!$ solution of this IVP on I.

Proof: $y' = \underbrace{-P(x)y + F(x)}_{F(x,y)}$. Apply Theorem 1.2.1 to this IVP:

$\frac{\partial F}{\partial y} = -P(x)$. Since $P(x)$ & $F(x)$ cont. on I $\Rightarrow$ $F(x,y)$ & $\frac{\partial F}{\partial y}$ cont. on I

$\Rightarrow \exists$ some interval I_0 with a unique solution. But by

$\boxed{*}$. any solution has the form $\circledast$ & $x_0 \in I \Rightarrow$ the solution is just a matter of plugging x_0 & y_0 into $\circledast$ & solving for c. i.e. $x_0 \Leftrightarrow$ distinct c.

$\therefore$ This interval I_0 of existence & uniqueness is the entire interval I.

2.4: Exact Eq^{n}'s:

Notation: If $z = f(x,y)$ has continuous first partials in a region $R = [a,b] \times [c,d]$, then its <u>differential</u> is $dz = \underbrace{\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy}_{\text{"} \nabla f \text{"}}$.

Def^{n}: A 1st-order DE of the form $M(x,y)dx + N(x,y)dy = 0$ is called an <u>exact eq^{n}</u> (in a region $R = [a,b]\times[c,d]$) if $M(x,y)dx + N(x,y)dy = \nabla f$ for some function $f(x,y)$ defined on R.

e.g.7 The DE $(2xy+6) + (x^2-3y^2)\frac{dy}{dx} = 0$ can be written in the form: $\underbrace{(2xy+6)}_{M(x,y)}\,dx + \underbrace{(x^2-3y^2)}_{N(x,y)}\,dy = 0$.

Consider the function $f(x,y) = x^2y + 6x - y^3$.

$\nabla f = (2xy+6)\,dx + (x^2-3y^2)\,dy = M(x,y) + N(x,y)$.

$\therefore$ The DE is exact.

For those who took Vector calc., same criterion as a v.f. being a gradient v.f..

Theorem 2.4.1 [<u>Criterion for Exact DE</u>]: If $M(x,y)$ & $N(x,y)$ are C^1 in $R = [a,b]\times[c,d]$, then $M(x,y)dx + N(x,y)dy = 0$ is exact $\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solving Exact DE's: To solve an exact DE, we want to find the F s.t. $\nabla F = M(x,y)\,dx + N(x,y)\,dy$. Then an implicit solution is given by $F(x,y) = c$.

Let's demonstrate how to find such an F via an example:

Example: Is $(5y - 2x)\,y' - 2y = 0$ exact? If so, solve it.

Rewrite as: $\underbrace{(5y-2x)}_{N(x,y)}dy \underbrace{-2y}_{M(x,y)}dx = 0$.

$\frac{\partial M}{\partial y} = -2 = \frac{\partial N}{\partial x} \Rightarrow$ exact by Theorem 2.4.1, since M & N are C^1 on $(-\infty,\infty)$.

$\therefore$ We know $\exists\ F(x,y)$ s.t. $\nabla F = M(x,y)\,dx + N(x,y)\,dy$.

i.e. $\Rightarrow \frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N$.

$\frac{\partial F}{\partial x} = M \Rightarrow \frac{\partial F}{\partial x} = -2y \Rightarrow F = \int -2y\,dx \Rightarrow F = -2yx + g(y)$.

F is a function of x & y, so integrating w.r.t. x means a "constant" has y's & real #'s.

$\frac{\partial F}{\partial y} = N \Rightarrow -2x + g'(y) = \underbrace{5y - 2x}_{N}$

$\Rightarrow \frac{dg}{dy} = 5y \Rightarrow g = \int 5y\,dy$

$\Rightarrow g = \frac{5}{2}y^2 + c$.

chose $c=0$... could choose c to be anything

$\therefore F = -2yx + \frac{5}{2}y^2$.

$\therefore$ A solution to the DE is $-2yx + \frac{5}{2}y^2 = C$.

Integrating Factors: Sometimes $\exists$ an eq^{n} of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, but we can multiply both sides by an appropriate function $\mu(x,y)$ s.t. $\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$ is exact. This function $\mu(x,y)$ is called an **integrating factor**.

- If $\dfrac{M_y - N_x}{N}$ is a function of x alone, then $\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$ is an integrating factor.

- If $\dfrac{N_x - M_y}{M}$ is a function of y alone, then $\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$ is an integrating factor.

By turning our non-exact eq^{n} into an exact one, we can solve this new exact eq^{n} & find an implicit solution for the original eq^{n}.

e.g.7 Solve $\underbrace{(x^2+y^2-5)}_{M}\,dx = \underbrace{(y+xy)}_{-N}\,dy$, $y(0)=1$.

This eq^{n} is not exact, since $\dfrac{\partial M}{\partial y} = 2y$ & $\dfrac{\partial N}{\partial x} = -y$. not same!

$$\frac{M_y - N_x}{N} = \frac{2y+y}{-(y+xy)} = \frac{-3y}{y(1+x)} = \frac{-3}{1+x} \text{ depends on } x \text{ alone.}$$

$$\therefore\ e^{\int \frac{-3}{x+1}\,dx} = e^{\ln\left((x+1)^{-3}\right)} = (x+1)^{-3}.$$

So, $\underbrace{(x+1)^{-3}(x^2+y^2-5)}_{\tilde{M}}\,dx\ \underbrace{-\,(y+xy)(x+1)^{-3}}_{\tilde{N}} = 0$ is exact.

Want to find an f s.t. $\nabla f = \tilde{M}\,dx + \tilde{N}\,dy$.

$$\frac{\partial F}{\partial y} = N \Rightarrow F = \int \frac{-y}{(x+1)^2}\,dy = \frac{-y^2}{2(x+1)^2} + h(x).$$

$$\frac{\partial F}{\partial x} = M \Rightarrow -\frac{y^2}{\cancel{2}} \cdot \frac{-\cancel{2}}{(x+1)^3} + h'(x) = \frac{x^2+y^2-5}{(x+1)^3}$$

$$\Rightarrow h'(x) = \frac{x^2-5}{(x+1)^3} \Rightarrow h(x) = \int \frac{x^2}{(x+1)^3} - \frac{5}{(x+1)^3}\,dx$$

[Using Partial Fractions, can write $\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$

$\Leftrightarrow x^2 = A(x+1)^2 + B(x+1) + C$

$\Leftrightarrow x^2 = Ax^2 + (2A+B)x + (A+B+C)$

$\Leftrightarrow A=1$ & $2A+B=0$ & $A+B+C=0$

$\Leftrightarrow A=1$ & $B=-2$ & $C=1$. $\therefore \frac{x^2}{(x+1)^3} = \frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{1}{(x+1)^3}$]

$$= \int \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3} - \frac{5}{(x+1)^3}\,dx$$

$$= \ln|x+1| + \frac{2}{x+1} - \frac{4}{2(x+1)^2} + C.$$

$$\therefore F(x,y) = \frac{-y^2}{2(x+1)^2} + \ln|x+1| + \frac{2}{x+1} - \frac{2}{(x+1)^2}.$$

$\therefore$ An implicit solution is $\frac{-y^2}{2(x+1)^2} + \ln|x+1| + \frac{2}{x+1} - \frac{2}{(x+1)^2} = c.$