6.3: Solutions About Singular Points

Def^{n}: A singular points $x = x_0$ of the eq^{n} $y'' + P(x)y' + Q(x)y = 0$ is said to be a **regular singular point** if the functions $p(x) = (x - x_0)P(x)$ & $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. A singular point that is not regular is said to be an **irregular singular point** of the eq^{n}.

i.e.) If $P(x)$ & $Q(x)$ are rational functions reduced to lowest terms $\left[P(x) = \frac{a_1(x)}{a_2(x)} \text{ \& } Q(x) = \frac{a_0(x)}{a_2(x)}\right]$ & if x_0 is a singular pt of the eq^{n} then x_0 is a regular singular pt if $(x - x_0)$ appears at most to the 1st power in the denominator of $P(x)$ & at most to the 2nd power in the denominator of $Q(x)$.

e.g.) Find the regular & irregular singular pts of the following:

a) $x^2(x-5)^2 y'' + 4xy' + (x^2 - 25)y = 0.$

$$P(x) = \frac{4x}{x^2(x-5)^2} = \frac{4}{x(x-5)^2} \Rightarrow x = 0 \text{ \& } x = 5 \text{ singular pts.}$$

$$Q(x) = \frac{(x^2 - 25)}{x^2(x-5)^2} = \frac{(x+5)(x-5)}{x^2(x-5)^2} = \frac{x+5}{x^2(x-5)}.$$

$x = 5$ irregular singular pt, b/c appears to the 2nd power in $P(x)$.

$x = 0$ regular singular pt, b/c in 1st power in $P(x)$ & 2nd power in $Q(x)$.

(b) $x(x+3)^2 y'' - y = 0$.

$P(x) = 0$. $Q(x) = \frac{-1}{x(x+3)^2}$. So, $x=0$ & $x=-3$ regular singular pts.

(c) $x(x^2+1)^2 y'' + y = 0$.

$P(x) = 0$. $Q(x) = \frac{1}{x(x^2+1)^2}$. So, $x=0$ & $x = \pm i$ regular singular pts.

Note: The solution is only a power series if r is a positive integer or zero. Otherwise, it's just an infinite series.

Theorem 6.3.1 [Frobenius' Theorem]: If $x = x_0$ is a regular singular pt of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then there exists at least one solution of the form:

$$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$$

where $c_0 \neq 0$ & where r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Goal: We want to find r & compute the coefficients c_n, $n \geq 1$, in terms of c_0.

Note: Like in the previous section, for the sake of simplicity we'll only find solutions about the pt $x=0$.
[If $x_0 \neq 0$, make the substitution $t = x - x_0$].

Method of Solution [Method of Frobenius]:

1. Substitute $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the DE.

2. Combine the series & set coefficients equal to zero.

3. Setting the coefficients containing c_0 equal to zero will give you two values for r [possibly a double root], r_1 & r_2, with $r_1 \geq r_2$.

4. Find a recurrence relation & take $r = r_1$ to determine $c_1, c_2, \ldots$ recursively in terms of c_0 & r_1.

5. A solution is given by $y = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$, where the a_n's are defined in terms of a_0 & r_1.

e.g. 7 Find a series solution about the regular singular point $x=0$ of $x^2 y'' - xy' + (1-x)y = 0$, $x>0$.

① $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$.

$$0 = x^2 y'' - xy' + (1-x)y = \sum_{n=0}^{\infty} \underbrace{(n+r)(n+r-1) c_n x^{n+r}}_{k=n} - \sum_{n=0}^{\infty} \underbrace{(n+r) c_n x^{n+r}}_{k=n}$$

$$+ \sum_{n=0}^{\infty} \underbrace{c_n x^{n+r}}_{k=n} - \sum_{n=0}^{\infty} \underbrace{c_n x^{n+r+1}}_{k=n+1}$$

② $$= x^r \left[\sum_{k=0}^{\infty} (k+r)(k+r-1) c_k x^k - \sum_{k=0}^{\infty} (k+r) c_k x^k + \sum_{k=0}^{\infty} c_k x^k - \sum_{k=1}^{\infty} c_{k-1} x^k \right]$$

$$= x^r \left[r(r-1) c_0 - r c_0 + c_0 + \sum_{k=1}^{\infty} \left[(k+r)(k+r-1) c_k - (k+r) c_k + c_k x^k - c_{k-1} \right] x^k \right]$$

$$\Rightarrow c_0 [r(r-1) - r + 1] = 0 \quad \& \quad [(k+r)(k+r-1) - (k+r) + 1] c_k - c_{k-1} = 0$$

$c_0 \neq 0$ $$\Rightarrow \underbrace{r^2 - 2r + 1 = 0}_{\text{indicial eq'n}} \quad \& \quad \underbrace{c_k = \frac{c_{k-1}}{(k+r)[k+r-2]+1}}_{\text{recurrence relation}} = \frac{c_{k-1}}{(k+r)^2 - 2(k+r) + 1} = \frac{c_{k-1}}{(k+r-1)^2}$$

③ $\Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$
$\Rightarrow r_1 = 1$.

$r = r_1 = 1$

④ $\Rightarrow c_k = \frac{c_{k-1}}{k^2}$

$\Rightarrow c_1 = c_0$ & $c_2 = \frac{1}{2!}c_0$ & $c_3 = \frac{1}{9 \cdot 2!} c_0$ & $c_4 = \frac{1}{4^2 \cdot 3^2 \cdot 2^2} c_0$

$\Rightarrow c_K = \frac{1}{(K!)^2} c_0$.

$\therefore$ A solution is given by $y = x \sum_{n=0}^{\infty} \frac{c_0}{(n!)^2} x^n = \sum_{n=0}^{\infty} \frac{c_0}{(n!)^2} x^{n+1}$.

- Where does this solution converge?

This is a power series, so by the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{c_0}{[(n+1)!]^2}}{\frac{c_0}{(n!)^2}} \right| = \lim_{n\to\infty} \left| \frac{(n!)^2}{[(n+1)!]^2} \right| = \lim_{n\to\infty} \left| \frac{(n \cdot n-1 \cdots 1)(n \cdot (n-1) \cdots 1)}{[(n+1) \cdot n \cdots 1][(n+1) \cdot n \cdots]} \right|$$

$= \lim_{n\to\infty} \left| \frac{1}{(n+1)^2} \right| = 0 \Rightarrow R = \infty$. But we made the assumption throughout our computation that $x > 0$, so the solution converges on the interval $(0, \infty)$.

Def^{n}: After substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the DE $y'' + P(x)y' + Q(x)y = 0$ & simplifying, the indicial eq^{n} is a quadratic eq^{n} in r that results from equating the total coefficient of the lowest power of x to zero.

- Knowing the indicial eq^{n} in advance can give us information about the nature of solutions.

Theorem: The indicial eq^{n} of $y'' + P(x)y' + Q(x)y = 0$ is $r(r-1) + a_0 r + b_0 = 0$, where $a_0 = \lim_{x\to 0} xP(x)$ & $b_0 = \lim_{x\to 0} x^2 Q(x)$. [Here we're still assuming $x=0$ is a regular singular pt].

Proof $x=0$ regular singular pt $\Rightarrow$ $xP(x)$ & $x^2Q(x)$ both analytic at $x=0$ $\Rightarrow$ they can be written as a power series centered at $x=0$:

$$xP(x) = \sum_{n=0}^{\infty} a_n x^n \quad \& \quad x^2Q(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Multiplying our DE by x^2 we have:

$x^2y'' + x[xP(x)]y' + x^2Q(x)y = 0$. Sub. $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into eq^{n}:

$$y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}.$$

$$0 = x^2y'' + x[xP(x)]y' + x^2Q(x) = \underbrace{\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r}}_{x^2y''}$$

$$+ \underbrace{\left[\sum_{n=0}^{\infty} a_n x^n\right]}_{xP(x)} \underbrace{\left[\sum_{n=0}^{\infty} c_n(n+r)x^{n+r}\right]}_{xy'} + \underbrace{\left[\sum_{n=0}^{\infty} b_n x^n\right]}_{x^2Q(x)} \underbrace{\left[\sum_{n=0}^{\infty} c_n x^{n+r}\right]}_{y}$$

$$= \left[c_0 r(r-1)x^r + \sum_{n=1}^{\infty} c_n(n+r)(n+r-1)x^{n+r}\right]$$

$$+ [a_0 + a_1x + a_2x^2 + \dots][c_0 r x^r + c_1(1+r)x^{r+1} + \dots] + [b_0 + b_1x + \dots][c_0x^r + c_1x^{r+1} + \dots]$$

$$\Rightarrow [c_0 r(r-1) + a_0c_0r + b_0c_0]x^r = 0$$

Lowest Term is x^r

$\Rightarrow$ $r(r-1) + a_0r + b_0 = 0.$

$x>0$ & $c_0 \neq 0$

$$\lim_{x\to 0} xP(x) = \lim_{x\to 0} \sum_{n=0}^{\infty} a_n x^n = a_0$$

$$\& \lim_{x\to 0} x^2Q(x) = \lim_{x\to 0} \sum_{n=0}^{\infty} b_n x^n = b_0. \quad \blacksquare$$

Theorem [Form of 2^{nd} Linearly Independent Solution]:

Suppose $x=0$ is a regular singular pt of $y''+P(x)y'+Q(x)y=0$ & that r_1 & r_2, the associated indicial roots, are real, with $r_1 \geq r_2$.

(a) If $r_1 \neq r_2$ & $r_1 - r_2$ is <u>not</u> a positive integer:

There exist 2 linearly independent solutions of the form
$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$, $c_0 \neq 0$, $y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$, $b_0 \neq 0$.

(b) If $r_1 \neq r_2$ & $r_1 - r_2$ is a positive integer:

There exist 2 linearly independent solutions of the form
$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$, $c_0 \neq 0$, $y_2 = c\, y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$, $b_0 \neq 0$, where c is a constant. [c could be zero].

(c) If $r_1 = r_2$: $\exists$ 2 linearly independent solutions of the form
$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$, $c_0 \neq 0$, $y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}$.

e.g. 7 $x^2 y'' - xy' + (1-x)y = 0$, $x > 0$.

We found [2 pg.'s ago] a solution to this DE: $y_1 = \sum_{n=0}^{\infty} \frac{c_0}{(n!)^2} x^{n+1}$.

We also found that the indicial eq'n had a double root $r_1 = r_2 = 1$.

$\therefore$ By this Theorem, there exists a 2^{nd} solution of the form: $y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n+1}$. [We could find these coefficients b_n if we wanted too... by plugging y_2, y_2', y_2'' into the DE... but it's long + tedious + we won't do so here. ...But if you did you'd get $y_2 = y_1 \ln x - 2x^2 - \frac{3}{4}x^3 - \frac{11}{108}x^4 + \ldots$

$\boxed{\text{e.g. 7}}$ $(x+2)x^2y'' - xy' + (1+x)y = 0$, $x > 0$. Describe the form of the solutions to this DE about the regular singular point $x=0$. Then find solutions [Find the first 4 nonzero terms of each].

Here $P(x) = \dfrac{-1}{x(x+2)}$ + $Q(x) = \dfrac{1+x}{x^2(x+2)}$.

$$a_0 = \lim_{x \to 0} xP(x) = \lim_{x \to 0} \frac{-1}{x+2} = \frac{-1}{2}.$$

$$b_0 = \lim_{x \to 0} x^2 Q(x) = \lim_{x \to 0} \frac{1+x}{x+2} = \frac{1}{2}.$$

$\therefore$ The indicial eq^n is $r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0$

$\Rightarrow r^2 - \frac{3}{2}r + \frac{1}{2} = 0 \Rightarrow 2r^2 - 3r + 1 = 0$. $\quad r^2 - 3r + 2 = 0$

$(2r-2)(2r-1)$

$r_1 - r_2 = 1 - \frac{1}{2} = \frac{1}{2}$ not an integer $\quad (r-1)(2r-1) \Rightarrow r_1 = 1, r_2 = \frac{1}{2}$.

$\Rightarrow \exists$ 2 lin. ind. solutions of the form $y_1 = \sum_{n=0}^{\infty} c_n x^{n+1}$, $c_0 \neq 0$

& $y_2 = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$, $b_0 \neq 0$.

$\boxed{\text{Find the 2 solutions:}}$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}.$$

$$0 = (x+2)x^2y'' - xy' + (1+x)y$$

$$= (x+2)\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} + (1+x)\sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= \underbrace{\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r+1}}_{k=n+1} + 2\underbrace{\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r}}_{k=n}$$

$$- \underbrace{\sum_{n=0}^{\infty} c_n(n+r)x^{n+r}}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+1}}_{k=n+1}$$

$$= x^r\Big[\sum_{k=1}^{\infty} c_{k-1}(k-1+r)(k-2+r)x^k + 2\sum_{k=0}^{\infty} c_k(k+r)(k+r-1)x^k$$

$$- \sum_{k=0}^{\infty} c_k(k+r)x^k + \sum_{k=0}^{\infty} c_k x^k + \sum_{k=1}^{\infty} c_{k-1}x^k\Big]$$

$$= x^r\Big[\big[2c_0 r(r-1) - c_0 r + c_0\big] + \sum_{k=1}^{\infty}\big[c_{k-1}(k-1+r)(k-2+r)$$

$$+ 2c_k(k+r)(k+r-1) - c_k(k+r) + c_k + c_{k-1}\big]x^k\Big]$$

$$\Rightarrow 2r(r-1) - r + 1 = 0 \ \ \& \ \ c_k\big[2(k+r)(k+r-1) - (k+r) + 1\big]$$

$$+ c_{k-1}\big[(k-1+r)(k-2+r) + 1\big] = 0.$$

$r=1$: $c_k\big[2(k+1)k - k\big] + c_{k-1}\big[k(k-1)+1\big] = 0$

$$\Rightarrow c_k = \frac{-k(k-1)-1}{2(k+1)k-k}\,c_{k-1} \Rightarrow c_1 = \frac{-1}{3}c_0$$

$$\& \ \ c_2 = \frac{-3}{10}\left(-\frac{1}{3}c_0\right) = \frac{1}{10}c_0 \ \ \& \ \ c_3 = \frac{-7}{21}\left(\frac{1}{10}c_0\right) = -\frac{1}{3}\left(\frac{1}{10}c_0\right) = \frac{-c_0}{30}.$$

$$\therefore y_1 = \sum_{n=0}^{\infty} c_n x^{n+1} = c_0 x - \frac{1}{3}c_0 x^2 + \frac{1}{10}c_0 x^3 - \frac{1}{30}c_0 x^4 + \dots$$

when $c_0=1$

$$= x - \frac{1}{3}x^2 + \frac{1}{10}x^3 - \frac{1}{30}x^4 + \dots.$$

$\underline{r_2 = \frac{1}{2}}$: $c_k = -c_{k-1} \dfrac{[(k-\frac{1}{2})(k-\frac{3}{2})+1]}{2(k+\frac{1}{2})(k-\frac{1}{2})-(k+\frac{1}{2})+1}$

$\Rightarrow c_1 = -\dfrac{[\frac{1}{2}(-\frac{1}{2})+1]}{2(\frac{3}{2})(\frac{1}{2})-\frac{3}{2}+1} c_0 = \dfrac{-\frac{3}{4}c_0}{1} = -\dfrac{3}{4} c_0$,

& $c_2 = \dfrac{3c_0}{4}\dfrac{[\frac{3}{2}(\frac{1}{2})+1]}{2(\frac{5}{2})(\frac{3}{2})-\frac{5}{2}+1} = \dfrac{3}{4}c_0 \dfrac{\frac{7}{4}}{6} = \dfrac{3}{4}\cdot\dfrac{7}{4}\cdot\dfrac{1}{6}c_0 = \dfrac{7}{32}c_0$,

& $c_3 = -\dfrac{7}{32}c_0 \dfrac{[\frac{5}{2}(\frac{3}{2})+1]}{2(\frac{7}{2})(\frac{5}{2})-\frac{7}{2}+1} = -\dfrac{7}{32}c_0 \dfrac{\frac{19}{4}}{\frac{35}{2}-\frac{5}{2}} = -\dfrac{7}{32}c_0\dfrac{19}{4}\cdot\dfrac{2}{30}$

$= -\dfrac{133}{1920}c_0$.

32
30
960
2
1920
619
7
133

$\therefore y_2 = \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}}$

$= c_0 x^{\frac{1}{2}} - \frac{3}{4}c_0 x^{\frac{3}{2}} + \frac{7}{32}c_0 x^{\frac{5}{2}} - \frac{133}{1920}c_0 x^{\frac{7}{2}} + \ldots$

$= x^{\frac{1}{2}} - \frac{3}{4}x^{\frac{3}{2}} + \frac{7}{32}x^{\frac{5}{2}} - \frac{133}{1920}x^{\frac{7}{2}} + \ldots$.

when $c_0 = 1$

- What is a general solution for this DE?

A general solution to this DE is

$y = C_1 y_1 + C_2 y_2 = c_1[x - \frac{1}{3}x^2 + \frac{1}{10}x^3 - \frac{1}{30}x^4 + \ldots]$

$+ c_2[x^{\frac{1}{2}} - \frac{3}{4}x^{\frac{3}{2}} + \frac{7}{32}x^{\frac{5}{2}} - \frac{133}{1920}x^{\frac{7}{2}} + \ldots]$.

- Exam Review: Tues. Aug. 4th, 2pm-4pm, BSB/137.
- Exam: Wed. Aug. 5th, 7pm-10pm, HH 104 (Bas-O)
  BSB 105 (Pen-Z).
- Office Hours: Fri. Jul. 31st, 12pm-1:30pm, Tues. Aug. 4th, 4pm-5:30pm