https://evals.mcmaster.ca.
Mon. Jul. 27th, 10am
– Fri. Jul. 31st, 4pm.

6.2: Solutions About Ordinary Points

In this section, we'll find solutions to 2^{nd}-order linear DE's with nonconstant coefficients.

Def^n: A point $x = x_0$ is said to be an <u>ordinary point</u> of $y'' + P(x)y' + Q(x)y = 0$ if both $P(x)$ & $Q(x)$ are analytic at x_0. A point that is <u>not</u> an ordinary point is said to be a <u>singular point</u> of the DE.

e.g.) $y'' + \underbrace{e^x}_{P(x)} y' + \underbrace{(\cos x)}_{Q(x)} y = 0.$

e^x & $\cos x$ are analytic for all values of x (i.e. they can be represented by their Taylor series at any point x_0). $\therefore$ Every x is an ordinary point.

e.g.) $a_2 y'' + a_1 y' + a_0 y = 0$: [homog. DE w/ constant coef.], $a_2 \neq 0$.

If $a_2 \neq 0$, then $P(x) = \frac{a_1}{a_2}$ & $Q(x) = \frac{a_0}{a_2}$. Constant functions are analytic for all $x \Rightarrow$ every x is an ordinary point.

e.g.) $y'' + \frac{1}{1-x} y' + 7y = 0$. Find an ordinary & singular point.

Here $P(x) = \frac{1}{1-x}$ & $Q(x) = 7$.

Recall: Rational functions are analytic everywhere, except where their denominator is zero. $\therefore$ $P(x)$ <u>not</u> analytic at $x = 1$ $\Rightarrow$ $x = 1$ singular point. All other points are ordinary points.

e.g.) $y'' + \frac{1}{1+x^2} y' + 7y = 0$. $x = \pm i$: singular point. * We can consider complex numbers too!

Theorem 6.2.1: [Existence of Power Series Solutions]:

If $x = x_0$ is an ordinary point of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, we can always find 2 linearly independent solutions in the form of a power series centered at x_0.

i.e.) $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$.

A power series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

This distance R is a lower bound for the radius of convergence. The solution y is called a solution about the ordinary point x_0.

e.g.) Find the minimum radius of convergence of a power series solution of $(x^2 - 2x + 10)y'' + xy' - 4y = 0$

a) about the ordinary point 0.

b) about the ordinary point 1.

Here $P(x) = \dfrac{x}{x^2 - 2x + 10}$ & $Q(x) = \dfrac{-4}{x^2 - 2x + 10}$.

$x = \dfrac{2 \pm \sqrt{4 - 4(10)}}{2} = 1 \pm 3i$. So, this eq'n has singular points $1+3i$ & $1-3i$.

Recall: The distance b/w 2 complex numbers $a_1 + b_1 i$ & $a_2 + b_2 i$ is given by $\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$.

a) $0 = 0 + 0i$, so the distance b/w 0 & the closest singular point is $\sqrt{3^2 + 1^2} = \sqrt{10}$. ∴ min. rad. of conv. is $R = \boxed{\sqrt{10}}$.

b) $1 = 1 + 0i$. The distance b/w 1 & $1 \pm 3i$ is: $\sqrt{(1-1)^2 + 3^2} = 3$.
$\therefore R = 3$ is the min. rad. of convergence.

Method of Solution [Finding a Power Series Solution] [of a homogeneous linear 2nd order DE]:

For the sake of simplicity, we'll only find power series solutions about the ordinary point $x = 0$.
[to find a solution about $x_0 \neq 0$, substitute $t = x - x_0$, find solutions of the form $y = \sum_{n=0}^{\infty} c_n t^n$, then resubstitute $t = x - x_0$].

1. Substitute $y = \sum_{n=0}^{\infty} c_n x^n$ into the DE.
[this is our guess for the solution, & we want to locate the c_n's. You'll have to compute y' & y'' here].

2. Combine the series. [i.e., in each series make a substitution $k = n + j$ & add the series together].

3. Equate all coefficients to the RHS of the eq^n to determine the coefficients c_n. [we use the Identity Property here].

4. This leads us to two distinct sets of coefficients, so that we have 2 distinct power series y_1 & y_2. Therefore, the general solution is $y = c_0 y_1 + c_1 y_2$.

e.g. 7) Find a general solution to $2y'' + xy' + y = 0$.

Here $P(x) = \frac{x}{2}$ & $Q(x) = \frac{1}{2}$, so all x's are ordinary points. In particular, $x = 0$ is an ordinary point, so by Theorem 6.2.1 we're guaranteed 2 linearly independent solutions in the form of a power series centered at 0.

$\boxed{1}$ $y=\sum_{n=0}^{\infty} c_n x^n$, $y'=\sum_{n=1}^{\infty} c_n n x^{n-1}$, $y''=\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$.

$$2y''+xy'+y = 2\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n$$

$\boxed{2}$
$$= \underbrace{\sum_{n=2}^{\infty} 2c_n n(n-1)x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty} 2c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2(2) + \left[\sum_{k=1}^{\infty} 2c_{k+2}(k+2)(k+1)x^k\right] + \left[\sum_{k=1}^{\infty} k c_k x^k\right] + c_0 + \left[\sum_{k=1}^{\infty} c_k x^k\right]$$

$$= 4c_2 + c_0 + \sum_{k=1}^{\infty}\left[2c_{k+2}(k+2)(k+1) + (k+1)c_k\right]x^k = 0$$

$\boxed{3}$ $\Rightarrow 4c_2 + c_0 = 0$ & $\underbrace{2c_{k+2}(k+2)(k+1) + (k+1)c_k = 0}_{\text{recurrence relation}}$

$\Rightarrow c_2 = -\frac{1}{4}c_0$ & $c_{k+2} = \dfrac{-c_k}{2(k+2)}$.

So, we have: c_0, c_1, $c_2 = -\frac{1}{\underbrace{4}_{2^2}}c_0$, $c_3 = \dfrac{-c_1}{2\cdot 3}$,

$c_4 = \dfrac{-c_2}{2\cdot 4} = \dfrac{1}{\underbrace{2}_{2^1}\cdot\underbrace{4\cdot 4}_{2^4}}c_0$, $c_5 = \dfrac{-c_3}{2\cdot 5} = \dfrac{c_1}{2\cdot 5\cdot \underbrace{2\cdot 3}_{2^2}}$,

$c_6 = \dfrac{-c_4}{2\cdot 6} = \dfrac{-c_0}{\underbrace{2\cdot 4\cdot 4\cdot 2}_{2^6}\cdot \underbrace{6}_{3!}}$, $c_7 = \dfrac{-c_5}{2\cdot 7} = \dfrac{-c_1}{3\cdot 5\cdot 7\cdot 2^3}$,

$c_8 = \dfrac{-c_6}{2\cdot 8} = \dfrac{c_0}{2^{10}\cdot 6} = \dfrac{c_0}{2^8\, 4!}$

We can see the pattern for the coefficients is:

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n}\, n!} \quad \& \quad c_{2n+1} = \frac{(-1)^n c_1}{2^n\,[1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)]}.$$

4 $\therefore$ Two linearly independent solutions are:

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\, n!} x^{2n} \quad [\text{taking } c_0 = 1 \;\&\; c_1 = 0]$$

$$\& \quad y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\,[1\cdot 3\cdot 5 \cdots (2n+1)]} x^{2n+1} \quad [\text{taking } c_0 = 0 \;\&\; c_1 = 1].$$

Do IVP Example Here

$\therefore$ A general solution is given by $y = a_1 y_1 + a_2 y_2$. Since our DE has no singular points, this solution converges on $(-\infty, \infty)$ [by Theorem 6.2.1].

Note: Sometimes it may be too difficult to identify a formula for the recurrence relation. In this case, you'll be asked to write down the first 3 or 4 terms in the solution.

Nonpolynomial coefficients: If the coefficients in our 2nd-order linear DE are not polynomials, then first write the coefficient in terms of its power series [we can do this since $P(x)$ & $Q(x)$ are both analytic], then solve as normal.

Nonhomogeneous Linear DE's: A point x_0 is an ordinary point of a nonhomogeneous linear DE $y'' + P(x)y' + Q(x)y = F(x)$ if $P(x)$, $Q(x)$, & $F(x)$ are analytic at x_0. Theorem 6.2.1

Note: If we just want the first few terms of y_1 & y_2, obtain y_1 by setting $c_0 = 1$ & $c_1 = 0$, & y_2 by setting $c_0 = 0, c_1 = 1$.

extends to these DE's. We solve them in the exact same way as the homogeneous case [just bring $F(x)$ to the other side: $y'' + P(x)y' + Q(x)y - F(x) = 0$], or at the end just equate coefficients.

e.g.) Find the first 4 nonzero terms of a power series expansion about $x=0$ of a general solution to the DE $y'' - xy' + 2y = \cos x$.

1. $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$.

$y'' - xy' + 2y = \cos x$

2. $\Leftrightarrow \underbrace{\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} c_n n x^n}_{k=n} + 2 \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$\Leftrightarrow \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1) x^k - \sum_{k=1}^{\infty} c_k k x^k + 2 \sum_{k=0}^{\infty} c_k x^k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$\Leftrightarrow 2c_2 + 2c_0 + \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1) - kc_k + 2c_k\right] x^k = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

3. $\Rightarrow 2c_2 + 2c_0 = 1$ & $6c_3 + c_1 = 0$ & ...etc.

$\Rightarrow c_2 = \frac{1}{2} - c_0$ & $c_3 = -\frac{1}{6} c_1$ &

$\Rightarrow y = \sum_{n=0}^{\infty} c_n x^n = \underbrace{c_0}_{1} + \underbrace{c_1 x}_{2} + \underbrace{\left(\tfrac{1}{2} - c_0\right) x^2}_{3} + \underbrace{\left(-\tfrac{1}{6} c_1\right) x^3}_{4} + \cdots$.

i.e.) $y = \left[\frac{1}{2} x^2 + \cdots\right] + c_0\left[1 - x^2 + \cdots\right] + c_1\left[x - \frac{1}{6} x^3 + \cdots\right]$, &

so $y_p = \frac{1}{2} x^2 + \cdots$, $y_1 = 1 - x^2 + \cdots$, $y_2 = x - \frac{1}{6} x^3 + \cdots$

& $y = y_p + y_1 + y_2$.

e.g.7: Consider $y'' + e^x y' - y = 0$. Find 2 power series solutions [First 3 terms of each] about $x=0$.

$y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$,

$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

So, $0 = y'' + e^x y' - y = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \left[\sum_{n=0}^{\infty} \frac{1}{n!} x^n\right]\left[\sum_{n=1}^{\infty} c_n n x^{n-1}\right]$

$- \sum_{n=0}^{\infty} c_n x^n$

$= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} c_n x^n + \left[1 + x + \frac{x^2}{2!} + \cdots\right]\left[c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots\right]$

$= \left[2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots\right] - \left[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right]$

[Since we just want first 3 terms, let's write out each series].

$+ \left[c_1 + (2c_2 + c_1)x + (3c_3 + 2c_2 + \frac{c_1}{2!})x^2 + \cdots\right] = 0$

$\Rightarrow 2c_2 - c_0 + c_1 = 0$ & $6c_3 - c_1 + (2c_2 + c_1) = 0$

& $12 c_4 - c_2 + 3c_3 + 2c_2 + \frac{1}{2} c_1 = 0$

$\Rightarrow c_2 = \frac{1}{2} c_0 - \frac{1}{2} c_1$ & $c_3 = -\frac{1}{3} c_2$ & $c_4 = -\frac{1}{24} c_1 - \frac{1}{12} c_2 - \frac{1}{4} c_3$.

When $c_0 = 1, c_1 = 0$: $c_2 = \frac{1}{2}$, $c_3 = -\frac{1}{6}$, $c_4 = -\frac{1}{12}\left(\frac{1}{2}\right) - \frac{1}{4}\left(-\frac{1}{6}\right)$

$= -\frac{1}{24} + \frac{1}{24} = 0$.

So, $y_1 = 1 + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \cdots$.

When $c_0 = 0, c_1 = 1$: $c_2 = -\frac{1}{2}$, $c_3 = \frac{1}{6}$, $c_4 = -\frac{1}{24} - \frac{1}{12}\left(-\frac{1}{2}\right) - \frac{1}{4}\left(\frac{1}{6}\right)$

$= -\frac{1}{24}$.

So, $y_2 = x - \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots$.

• What is the minimum radius of convergence for y_1 & y_2?

Here $P(x) = e^x$ & $Q(x) = -1$. e^x & -1 have radius of convergence $(-\infty, \infty)$ $\Rightarrow$ $P(x)$ & $Q(x)$ are analytic at all points $\Rightarrow$ the DE has no singular points $\Rightarrow$ the minimum radius of convergence is $R = \infty$.

e.g.) Solve the IVP $2y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

From the 1st example we know we have the recurrence relations:

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n}\, n!} \quad \& \quad c_{2n+1} = \frac{(-1)^n c_1}{2^n [1 \cdot 3 \cdot 5 \cdots (2n+1)]}.$$

$y(0) = 1 \Rightarrow \sum_{n=0}^{\infty} c_n x^n = 1 \Rightarrow c_0 = 1$.

$y'(0) = 0 \Rightarrow \sum_{n=1}^{\infty} n c_n x^{n-1} = 0 \Rightarrow c_1 = 0 \Rightarrow$ all odd terms are zero.

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\, n!} x^{2n} = e^{-x^2/4}.$$

[Since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \Rightarrow e^{-\frac{x^2}{4}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-x^2}{4}\right)^n$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!\, 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!\, 2^{2n}}$].