1.1: Defⁿ's & Terminology:

Math 2C03 - Class #1

Jun. 22nd/15

Background: In each of the following, what is the independent variable & which is the dependent variable.

ⓐ $y(x) = x^2$. ⇝ x independent, y dependent.

ⓑ $\underbrace{\frac{dx}{dt}}_{x'} = 2$. ⇝ x dependent, t independent.

ⓒ $\underbrace{\frac{\partial^2 y}{\partial x^2}}_{y_{xx}} + \underbrace{\frac{\partial y}{\partial t}}_{y_t} = y + x$. ⇝ x & t independent, y dependent.

Notation:

$$\frac{dy}{\partial x} = y_x = y'.$$

$$\frac{d^2y}{\partial x^2} = y_{xx} = y''.$$

$$\vdots$$

$$\frac{d^n y}{\partial x^n} = \underbrace{y_{x \cdots x}}_{n \text{ times}} = y^{(n)}.$$

Note: Dependent variables should be thought of as functions.
e.g.⇒ $y(x) = x^2$ is sometimes written: $y = x^2$ when it is clear y is dependent.
i.e.⇒ $y: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$.

• **Defⁿ:** An eqⁿ containing the derivatives of one or more unknown functions (dependent variables) w.r.t. a single independent variable is called an ordinary differential eqⁿ (ODE).

e.g.⇒ $\frac{dy}{dx} = x^3$ is an ODE. Its solution is:

$$y = \int x^3\,dx = \tfrac{1}{4}x^4 + c. \text{ (on } (-\infty,\infty)).$$

★ e.g. (a) $y''' + x^8 y'' + y' = 0$, (b) $\frac{d^2y}{dx} + 7y - \left(\frac{dy}{dx}\right)^5 = 8$,

(c) $\frac{dx}{dt} + \frac{d^{(4)}y}{\partial t^4} = t^3 + xy$ are all examples of ODE's.

• **Def^n:** An eq^n involving partial derivatives of one or more unknown functions of two or more independent variables is called a partial differential eq^n (PDE).

e.g. $\frac{d^2y}{\partial x^2} + \frac{dy}{\partial t} = 8$ is a PDE. (it has 2 independent variables x & t).

* In this course, we will only consider ODE's. Therefore, instead of writing ODE, I'll write DE. *

• **Def^n:** The order of a DE is the order of the highest derivative in the eq^n.

e.g. In ★: (a) third-order, (b) 2^{nd}-order, (c) 4^{th}-order.

• **Notation:** An n^{th}-order DE in one dependent variable is often expressed as $F(x, y, y', \dots, y^{(n)}) = 0$, where $F: \mathbb{R}^{n+2} \to \mathbb{R}$.

• **Def^n:** An n^{th}-order DE $F(x, y, y', \dots, y^{(n)}) = 0$ is linear if F is linear in the variables $y, y', \dots, y^{(n)}$.

i.e. $F(x, y, y', \dots, y^{(n)}) = a_n(x)\frac{d^n y}{\partial x^n} + a_{n-1}(x)\frac{d^{n-1}y}{\partial x^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y - g(x) = 0$.

• **Def^n:** A DE is nonlinear is just a DE that is not linear.

e.g.: (a) $x^3 + \frac{d^2y}{dx^2} + xy + 2 = 0$. ⇝ Linear

(b) $x^2 y + \left(\frac{dy}{dx}\right)^3 = 0$. ⇝ Nonlinear, b/c $\left(\frac{dy}{dx}\right)^3$.

(c) $\sin y + \frac{dy}{dx} = 0$. ⇝ Nonlinear, b/c $\sin y$.

(d) $y\frac{dy}{dx} + x = 0$. ⇝ Nonlinear, b/c $y\frac{dy}{dx}$.

• Def'n: A function $y: \mathbb{R} \to \mathbb{R}$ is called $\underline{C^n}$ on an interval if it possesses at least n derivatives that are continuous on I.

e.g.: • Any polynomial is C^n, since polynomials are continuous, & the derivative of a polynomial is a polynomial.

• $y = 1/x$ is continuous at all points except zero.
i.e.: it's C^0 on $(0,\infty)$ & $(-\infty,0)$.
Its n^{th} derivative is $\frac{c}{x^n}$ for some constant c.
∴ $1/x$ is C^n on any interval not containing zero.

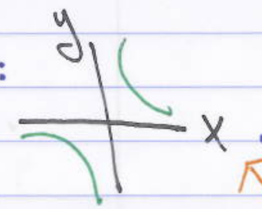
• Def'n: A solution of an nth-order DE (a.k.a. explicit solution) on an interval I, is any function y, C^n on I, which when substituted into the DE reduces the eq'n to an identity.
i.e.: $F(x, y, y', \dots, y^n) = 0 \;\; \forall \; x \in I$.

e.g.: Consider the DE $y' = y\cos x$.

$y = ce^{\sin x}$ is an explicit solution on $(-\infty,\infty)$, since $y' = \cos x \, ce^{\sin x}$ is continuous on $(-\infty,\infty)$ & $y\cos x = c e^{\sin x}\cos x = y'$. ✓

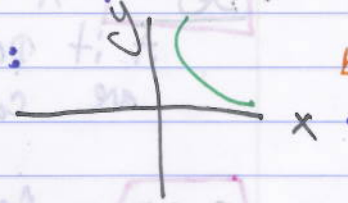
Notice: A solution includes 2 things: $\boxed{1}$ a function y
$\boxed{2}$ an interval I.

e.g.: • The graph of $y = 1/x$ looks like:



• The DE $xy' + y = 0$ has $y = 1/x$ as a solution on $(0, \infty)$, since $y' = -\frac{1}{x^2}$ cont. on $(0,\infty)$ & $xy' + y = x\left(-\frac{1}{x^2}\right) + \frac{1}{x} = 0$. ✓

The graph of the solution is:



not same

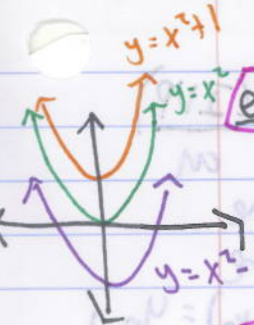
• **Def$^{\text{n}}$:** An <u>implicit solution</u> of a DE on an interval I is a relation $G(x,y) = 0$ s.t. $\exists$ at least one function y that satisfies both the DE & $G(x,y) = 0$.

e.g. 7 Consider the DE $\frac{dy}{dx} = \frac{-x}{y}$.

$x^2 + y^2 = 25$ is an implicit solution on $(-5,5)$, since $y = \sqrt{25 - x^2}$ satisfies $x^2 + y^2 = 25$ & the DE:

$\frac{dy}{dx} = \frac{\frac{1}{2}(-2x)}{\sqrt{25-x^2}} = \frac{-x}{y}$. ✓ Here $y = \sqrt{25 - x^2}$ is an explicit solution on $(-5,5)$.

• **Def$^{\text{n}}$:** A solution of a DE containing one arbitrary constant c is called a <u>one-parameter family of solutions</u>. A solution with n arbitrary constants is an <u>n-parameter family</u>. A solution with no arbitrary constants is a <u>particular solution</u>.



e.g. The DE $\frac{dy}{dx} = 2x$ has a one-parameter family of solutions $y = x^2 + c$. $y = x^2+1$ would be a particular solution. (both solutions on $(-\infty, \infty)$).

• **Def^n:** A singular solution of a DE is a solution which is not a member of a family of solutions of the eq^n.

e.g. Consider the DE $\frac{dy}{dx} = xy^{1/2}$. $y = (\frac{1}{4}x^2 + c)^2$ is a family of solutions of the DE on $(-\infty, \infty)$, since y is C^1 & $\frac{dy}{dx} = 2(\frac{1}{4}x^2+c)\cdot(\frac{1}{2}x) = xy^{1/2}$. ✓

But $y = 0$ is also a solution, since $\frac{dy}{dx} = 0 = x(0)^{1/2}$. However, there's no value of c which makes $y = (\frac{1}{4}x^2 + c)^2$ equal to zero. ∴ 0 is a singular solution.

• **Def^n:** • A system of DE's is 2 or more eq^n's involving derivatives of 2 or more unknown functions of a single independent variable.

• A solution of such a system of n eq^n's are n functions $y_1(x), \ldots, y_n(x)$ C^1 on a common interval I, that satisfy the system on this interval.

e.g. $\begin{cases} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 5x + 3y \end{cases}$ is a system of 2 DE's with solution

$\begin{cases} x = e^{-2t} + 3e^{6t} \\ y = -e^{-2t} + 5e^{6t} \end{cases}$ defined on $(-\infty, \infty)$.

Indeed, $\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} = x + 3y$. ✓

$\frac{dy}{dt} = 2e^{-2t} + 30e^{6t} = 5x + 3y$. ✓

Both x & y C^1. ✓

1.2: IVP's: An n^{th}-order initial value problem (IVP) is the problem of solving an n^{th} order DE $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$ on some interval I containing x_0, subject to the conditions $y(x_0) = y_0$, $y'(x_0) = y_1$, ..., $y^{(n-1)}(x_0) = y_{n-1}$ for some constants $y_i \in \mathbb{R}$.

e.g. $n=1$: Solve the IVP $\begin{cases} y' = y\cos x \\ y(0) = 2. \end{cases}$

$y = ce^{\sin x}$ on $(-\infty, \infty)$ solves DE, since $y' = c\cos x\, e^{\sin x} = y\cos x$.

$y(0) = 2 \Rightarrow 2 = ce^0 \Rightarrow c = 2$.

$\therefore$ solution to IVP is $y = 2e^{\sin x}$ on $(-\infty, \infty)$.

e.g. $n=2$: One could show $y = c_1e^x + c_2e^{-x}$ is a 2-parameter family of solutions of $y'' - y = 0$. Find a solution to the IVP $y'' - y = 0$, $y(0) = 1$, $y'(0) = 2$.

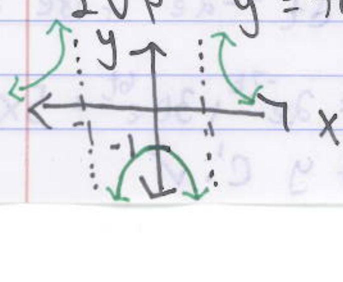
$y' = c_1e^x - c_2e^{-x}$.

$$\left.\begin{aligned} y(0) = 1 &\Rightarrow 1 = c_1 + c_2. \\ y'(0) = 2 &\Rightarrow 2 = c_1 - c_2. \end{aligned}\right\} \Rightarrow \begin{aligned} \tfrac{3}{2} &= c_1 \\ c_2 &= -\tfrac{1}{2}. \end{aligned}$$

$\therefore y = \tfrac{3}{2}e^x - \tfrac{1}{2}e^{-x}$ is a solution on $(-\infty, \infty)$.

e.g. Suppose the graph below is the graph of a function $y(x)$. Suppose also that $y(x)$ satisfies the DE $y' = F(x, y)$.

[a] Give an interval where the solution of the IVP $y' = F(x, y)$, $y(0) = -1$ is defined.



[b] Give 3 intervals where the solution of the DE $y' = F(x, y)$ is defined.

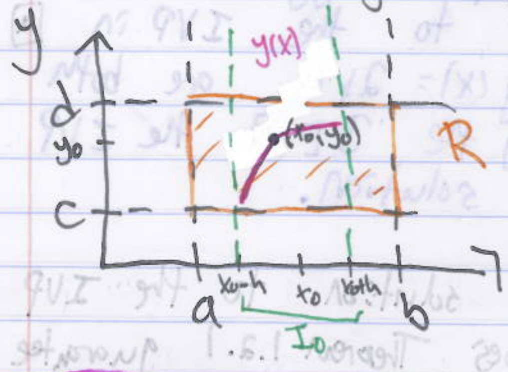
[a] The interval must contain $(0,-1)$, so $(-1,1)$ would be the largest interval, & any smaller interval in $(-1,1)$ would do the job too.

[b] We could choose $(-\infty,-1)$ or $(-1,1)$ or $(1,\infty)$ or any smaller intervals in those 3.

[Q] When does a solution to a First-order IVP exist & when is such a solution unique?

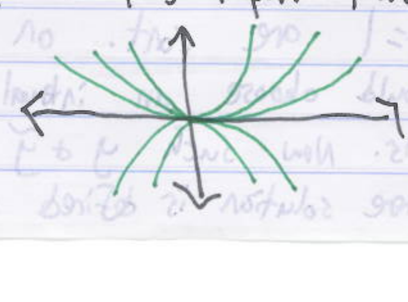
Theorem 1.2.1: Existence of a Unique Solution:

Let $R = [a,b] \times [c,d]$ contain the point (x_0, y_0) in its interior. If $f(x,y)$ & $\frac{\partial f}{\partial y}$ are continuous on $R \Rightarrow \exists$ some interval $I_0: (x_0-h, x_0+h)$, $h>0$ contained in $[a,b]$ & a unique function $y(x)$ defined on I_0 s.t. $y(x)$ is a solution of the IVP $y' = f(x,y)$, $y(x_0) = y_0$.



Example: Consider the DE $y' = \frac{2y}{x}$. The function $y(x) = cx^2$ satisfies this DE ($y' = 2cx = \frac{2y}{x}$).

The graph of this 1-par. family looks like:



$\boxed{a}$ Does the IVP $y' = 2y/x$, $y(0) = 0$ have a unique solution?

$\boxed{b}$ Does the IVP $y' = 2y/x$, $y(x_0) = y_0$ for $x_0 \neq 0$ have a unique solution?

In the notation of Theorem 1.2.1, here

$F(x,y) = \frac{2y}{x}$, $\frac{\partial F}{\partial y} = 2/x$. These functions are

cont. on any $R = [a,b] \times [c,d]$ not containing 0.

$\therefore$ By Theorem 1.2.1, if $y(x_0) = y_0$ for $x_0 \neq 0$, $\exists$ some interval $I_0 : (x_0 - h, x_0 + h)$ s.t. the IVP in $\boxed{b}$ has a unique solution.

Since $x = 0$ fails Theorem 1.2.1, this Theorem tells us nothing in $\boxed{a}$. However, looking at our 1-par. family, the graph suggests there are infinitely many solutions to the IVP in $\boxed{a}$. Indeed, $y(x) = x^2$ & $y(x) = 2x^2$ are both s.t. $y(0) = 0$ & satisfy the DE $\Rightarrow$ the IVP in $\boxed{a}$ has no unique solution.

25. Verify that the piece-wise-defined function

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

is a solution of the DE $xy' - 2y = 0$ on $(-\infty, \infty)$.

The DE is 1^{st}-order, so must show y satisfies DE & is C^1 on $(-\infty, \infty)$.

$\underline{x<0}$: $\frac{d}{dx}(-x^2) = -2x$. $xy' - 2y = x(-2x) - 2(-x^2) = -2x^2 + 2x^2 = 0.$ ✓

$\underline{x \geq 0}$: $\frac{d}{dx}(x^2) = 2x$. $xy' - 2y = x(2x) - 2(x^2) = 0.$ ✓

Need to check if y' continuous on $(-\infty, \infty)$.

$$y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0. \end{cases} \qquad \lim_{x \to 0^-} y' = \lim_{x \to 0} -2x = 0.$$

$$\lim_{x \to 0^+} y' = \lim_{x \to 0} 2x = 0.$$

same ✓

$\therefore$ DE C^1 on $(-\infty, \infty) \Rightarrow y$ is a solution on $(-\infty, \infty)$.

26. In Ex. 5 we saw $y = \sqrt{25 - x^2}$ & $y = -\sqrt{25 - x^2}$ are solutions of $y' = -\frac{x}{y}$ on $(-5, 5)$. Explain why

$$y = \begin{cases} \sqrt{25 - x^2} & -5 < x < 0 \\ -\sqrt{25 - x^2} & 0 \leq x < 5 \end{cases}$$

not a solution to DE on $(-5, 5)$.

$$y' = \begin{cases} \frac{1}{2}(25 - x^2)^{-\frac{1}{2}} \cdot -2x & -5 < x < 0 \\ -\frac{1}{2}(25 - x^2)^{-\frac{1}{2}} \cdot -2x & 0 \leq x < 5 \end{cases}$$

$$= \begin{cases} -x(25 - x^2)^{-\frac{1}{2}} & -5 < x < 0 \\ x(25 - x^2)^{-\frac{1}{2}} & 0 \leq x < 5. \end{cases}$$

Not necessary ✓!

We can see y is not cont. @ 0:

$\lim_{x \to 0^-} y = 5.$

$\lim_{x \to 0^+} y = -5.$

not same

Recall:
Differentiable at $x \Rightarrow$ cont. @ x.
$\therefore$ not cont @ $x \Rightarrow$ not diff. @ x.

y not cont. @ $x = 5 \Rightarrow y$ not diff. @ $x = 0$
$\Rightarrow y'$ not defined @ $x = 0$.
$\therefore$ y is not a solution on $(-5, 5)$, b/c y not C' on this interval.

43. Given that $y = \sin x$ is an explicit solution of the 1st-order DE $y' = \sqrt{1-y^2}$, Find an interval of defn.

We need to find an interval where y is C', & the DE is satisfied.
$y = \sin x$ defined everywhere, & $y' = \cos x$ cont..

$$y' = \sqrt{1-y^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$$

$y = \sin x \Rightarrow y' = \cos x$. So, $\cos x = |\cos x|$ when $\cos x > 0$.
A possible interval would be $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$\boxed{46.}$ The given graph represents the graph of an implicit solution $G(x,y) = 0$ of a DE $y' = f(x,y)$. The relation $G(x,y) = 0$ implicitly defines several solutions of the DE. Mark off segments of the corr. to graphs of solutions. Estimate an interval of defn of each solution ϕ.

We need each ϕ to be a $\underbrace{\text{function}}_{\phi(x) = y \ldots}$ & $\underbrace{\text{differentiable}}_{\text{can't have slope of } \infty}$.
each x maps to one y value.