Mat 202 - Tutorial #9

1. [11.1] Let G be the graph with vertex set [12] in which vertices u, v are adjacent $\Leftrightarrow$ u & v are relatively prime. Count the edges of G.

12 12
/\ /\
6 2 4 3

12
/\
2^2 3

Recall: Two integers are relatively prime when they have no common factor greater than 1.

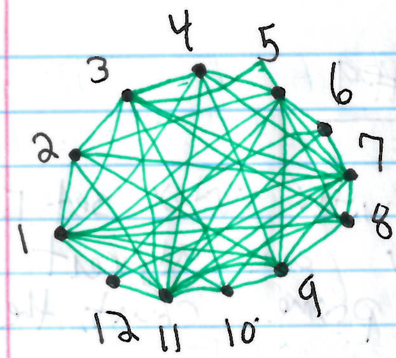
1: relatively prime to 2 - 12 (I'm going to use the convention that 1 is not relatively prime to itself... b/c we usually deal w/ simple graphs (no loops or multiple edges)).

12: rel. prime to 1, 5, 7, 11
11: rel. prime to 1 - 10, 12
10: " " " 1, 3, 7, 9, 11
9: " " " 1, 2, 4, 5, 7, 8, 10, 11
8: " " " 1, 3, 5, 7, 9, 11
7: " " " 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12
6: " " " 1, 5, 7, 11
5: " " " 1, 2, 3, 4, 6, 7, 8, 9, 11, 12
4: " " " 1, 3, 5, 7, 9, 11
3: " " " 1, 2, 4, 5, 7, 8, 10, 11
2: " " " 1, 3, 5, 7, 9, 11

[degree is # of edges a vertex is incident to].

Recall: # edges $= \frac{1}{2} \sum_{\text{vertices}} \deg(v)$. [Theorem 11.14]

If we total up the degree of each vertex (#) we get 90 $\therefore$ There are $\frac{90}{2} = 45$ edges.



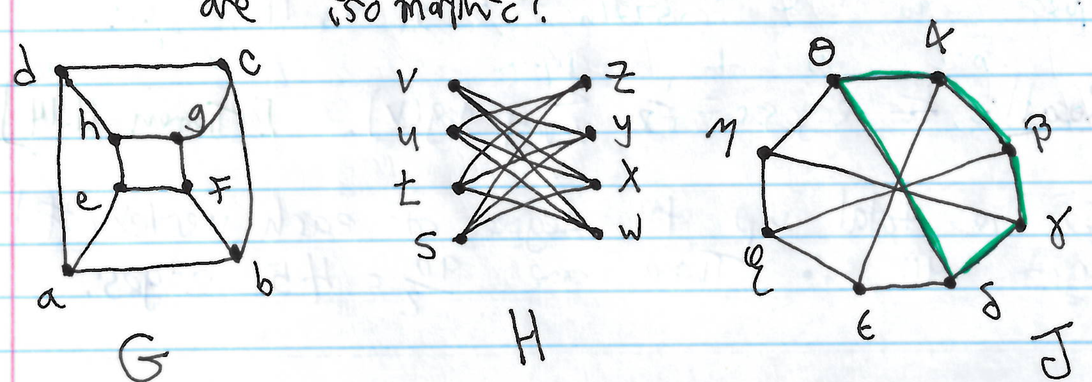
2. [11.5] Can the vertices of a simple graph all have distinct degrees.

Suppose our graph G has n vertices. Since it is simple, the degree of each vertex must be between 0 and n-1; i.e.⇒ there are only n distinct degrees.

In order to derive a contradiction, suppose G has the property that all of its vertices have a distinct degree ⇒ ∃ a vertex v w/ degree 0 and another vertex w with degree n-1.
But then w must be adjacent to all other n-1 vertices ⇒ w is adjacent to v ⇒ deg(v) > 0. ↯

∴ No, the vertices of a simple graph cannot all have distinct degrees.

3. [11.12] Among the graphs below, which pairs are isomorphic?



Recall: Two simple graphs G & H are called isomophic if there exist a bijection $F: V(G) \to V(H)$ between their list of vertices s.t. $uv \in E(G) \iff F(u)F(v) \in E(H)$. We write $G \cong H$.

Strategy: In general, it is difficult to tell whether two graphs are isomophic. The First thing you should do when asked whether or not they are isomorphic is to look For differences b/w the 2 graphs (to prove they are not isomophic): e.g.,

- # vertices
- # of edges
- list the degree of each vertex
- examine subgraphs.
- complement of the graph $\bar{G}$ ($G \cong H \iff \bar{G} \cong \bar{H}$)

  etc.

In this example, we notice that J has a subgraph which is a cycle of length 5, but G & H do not have such a subgraph. $\therefore J$ is not isomorphic to G or H.

We look at G & H and can't see any structural differences... we suspect that they may be isomophic, so we construct a bijection b/w vertices & check the adjacency relation:

$F: G \to H$

$a \mapsto s$	$d \mapsto z$	$g \mapsto w$
$b \mapsto x$	$e \mapsto y$	$h \mapsto t$
$c \mapsto u$	$f \mapsto v$	

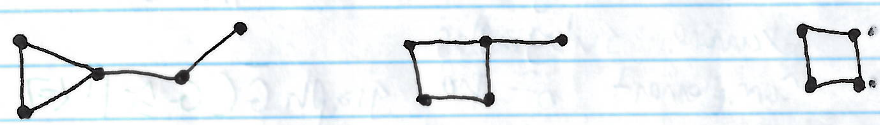
To check that this satisfies the adjacency relation, we must list all edge pairs and show that $(u,v) \in E(G) \iff (f(u), f(v)) \in E(H)$.

$(a,b) \leftrightarrow (s,x)$ ✓	$(c,g) \leftrightarrow (u,w)$ ✓
$(a,d) \leftrightarrow (s,z)$ ✓	$(d,h) \leftrightarrow (z,t)$ ✓
$(a,e) \leftrightarrow (s,y)$ ✓	$(h,e) \leftrightarrow (t,y)$ ✓
$(b,c) \leftrightarrow (x,u)$ ✓	$(h,g) \leftrightarrow (t,w)$ ✓
$(b,f) \leftrightarrow (x,v)$ ✓	$(e,f) \leftrightarrow (y,v)$ ✓
$(c,d) \leftrightarrow (u,z)$ ✓	$(f,g) \leftrightarrow (v,w)$ ✓

$\therefore G \cong H$.

4. Which pairs of graphs are isomorphic?

$\boxed{a}$



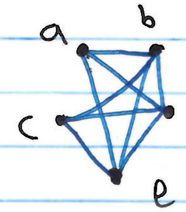
G H J

Obviously J is not isomorphic to G or H b/c it only has 4 vertices, but G & H have 5 vertices.

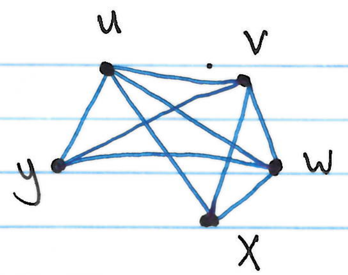
The vertices of G & H have degrees: $(3,2,2,2,1)$.

The only degree 3 vertex in G is adjacent to 3 deg. 2 vertices, whereas the only deg. 3 vertex in H is adjacent to a deg. 1 vertex and 2 deg. 2 vertices. $\therefore$ G & H cannot be isomorphic.

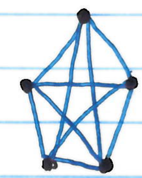
$\boxed{b}$



G



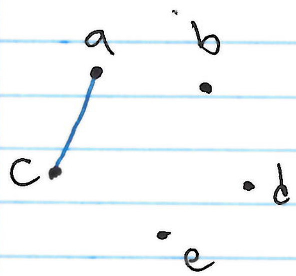
H



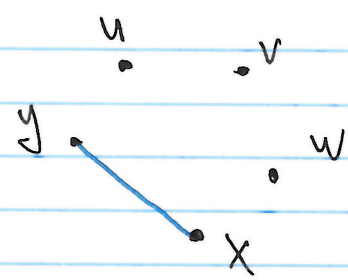
J

J <u>not</u> isomorphic to G or H b/c all vertices of J have deg. 4, whereas G & H both have vertices of deg. 3.

G & H certainly look isomorphic. Writing down an isomorphism is a little tedious... when each vertex has a high degree, it's often faster to use the fact that $G \cong H \iff \overline{G} \cong \overline{H}$ (Exercise 11.13) to prove they're isomorphic:



$\overline{G}$



$\overline{H}$

<u>Recall</u>: The <u>complement</u> $\overline{G}$ of a simple graph G is the graph w/ vertex set $V(G)$ and edge set $\{(u,v) \mid uv \notin E(G)\}$.

$F: \overline{G} \to \overline{H}$

$a \mapsto y$

$c \mapsto x$

$b \mapsto u$

$d \mapsto v$

$e \mapsto x$.

$(a,c) \mapsto (F(a), F(c)) = (y,x)$. ✓

$\therefore \overline{G} \cong \overline{H}$, and hence $G \cong H$.