Mat 20a: Tutorial #7

1. Compute: [a] $3^{1998} \bmod 4$ [b] $4^{26} \bmod 14$ [c] $2^{100} \bmod 13$.

Recall: [Fermat's Little Theorem]: If p prime & a not multiple of p, then $a^{p-1} \equiv 1 \bmod p$.

Recall: If $r \bmod n \equiv a$ and $s \bmod n \equiv b$, then $a+b \equiv s+r \bmod n$ and $a \cdot b = r \cdot s \bmod n$.

[a] $3^{1998} \bmod 4 = (3^2)^{999} \bmod 4 \equiv (1)^{999} \bmod 4 \equiv 1 \bmod 4$.

$3^2 = 9 \equiv 1 \bmod 4$

[b] $4^{26} \bmod 14$

$4^2 = 16 \equiv 2 \bmod 14$

$4^3 \bmod 14 = 4^2 \cdot 4 \bmod 14$
$\equiv 8 \bmod 14$

$4^4 \bmod 14 \equiv 32 \bmod 14$
$\equiv 4 \bmod 14$

$4^5 \bmod 14 \equiv 16 \bmod 14$
$\equiv 2 \bmod 14$.

$4^1 \equiv 4$ $4^4 \equiv 4$
$4^2 \equiv 2$ $4^5 \equiv 2$ ... etc.
$4^3 \equiv 8$ $4^6 \equiv 8$

So, $4^K \bmod 14 \equiv 4$ if $K \equiv 1 \bmod 3$
$\equiv 2$ if $K \equiv 2 \bmod 3$
$\equiv 8$ if $K \equiv 0 \bmod 3$.

$26 \equiv 2 \bmod 3 \Rightarrow 4^{26} \bmod 14 \equiv 2$.

[c] $2^{100} \bmod 13$. Here $p = 13$ prime and $a = 2$ not a multiple of 13. So, by Fermat's Little Theorem:

$2^{12} \equiv 1 \bmod 13$.

$2^{100} \bmod 13 = 2^{96} \cdot 2^4 \bmod 13 = (2^{12})^8 \cdot 2^4 \bmod 13 \equiv (1)^8 \cdot 2^4 \bmod 13$

$\equiv 16 \bmod 13 \equiv 3 \bmod 13$.

$12 \mid 96$, $96 = 8 \cdot 12$

2. Show that n^2-1 is always divisible by 3 provided that n itself is an integer not divisible by 3.

WTS $n^2-1 \equiv 0 \bmod 3$. If n is not divisible by 3, then either (i) $n \equiv 1 \bmod 3$ or (ii) $n \equiv 2 \bmod 3$.

Case (i): $n \equiv 1 \bmod 3 \Rightarrow n^2-1 \bmod 3 \equiv (1)^2-1 \bmod 3 \equiv 0 \bmod 3$.

Case (ii): $n \equiv 2 \bmod 3 \Rightarrow n^2-1 \bmod 3 \equiv 2^2-1 \bmod 3 = 3 \bmod 3 \equiv 0 \bmod 3$.

3. Given that $n \equiv 2 \bmod 7$, is $5n^3-2^n \equiv 5\cdot 2^3-2^2 \bmod 7$?

$5\cdot 2^3-2^2 \bmod 7 \equiv 36 \bmod 7 \equiv 1 \bmod 7$.

Consider $n=9$. $9 \equiv 2 \bmod 7$.

$5n^3-2^n \bmod 7 = 5(9)^3-2^9 \bmod 7 \equiv 5(2)^3-2^9 \bmod 7$

$2^2=4$
$2^3=8\equiv 1 \bmod 7$ $\equiv 40-2^9 \bmod 7 \equiv 5-2^9 \bmod 7 \equiv 5-(2^3)^3 \bmod 7 \equiv 5-(1)^3 \bmod 7$

$\equiv 5-1 \bmod 7 = 4 \bmod 7$.

So, when $n=9$, $n \equiv 2 \bmod 7$, but $5n^3-2^n \equiv 4 \bmod 7$

$\not\equiv 1 \bmod 7$. $\therefore$ This is not true.

13. Prove that n^3+5n is divisible by 6 for every $n \in \mathbb{N}$.

WTS $n^3+5n \equiv 0 \bmod 6$.

It suffices to show that n^3+5n is divisible by both 2 & 3 (i.e., $n^3+5n \equiv 0 \bmod 2$ & $n^3+5n \equiv 0 \bmod 3$).

Either $n \equiv 0 \bmod 2$ or $n \equiv 1 \bmod 2$.

If $n \equiv 0 \bmod 2 \Rightarrow n^3 + 5n \bmod 2 \equiv 0^3 + 5(0) \bmod 2 \equiv 0$.
If $n \equiv 1 \bmod 2 \Rightarrow n^3 + 5n \bmod 2 \equiv 1^3 + 5(1) \bmod 2 \equiv 6 \bmod 2 \equiv 0 \bmod 2$.

Either $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$, or $n \equiv 2 \bmod 3$.

If $n \equiv 0 \bmod 3 \Rightarrow n^3 + 5n \equiv 0 \bmod 3$.
If $n \equiv 1 \bmod 3 \Rightarrow n^3 + 5n \equiv 1^3 + 5 \bmod 3 \equiv 0 \bmod 3$.
If $n \equiv 2 \bmod 3 \Rightarrow n^3 + 5n \equiv 2^3 + 5(2) \bmod 3 \equiv 8 + 10 \bmod 3 \equiv 0 \bmod 3$.

$\therefore n^3 + 5n$ divisible by $2 \cdot 3 \Rightarrow n^3 + 5n$ is divisible by 6.

14. Show that the last digit of a nonnegative integer n is its remainder when divided by 10. Then determine the last digit in the decimal expansion of 4^{100}.

Suppose n has K digits. i.e. $n = n_1 n_2 \cdots n_K$

$= n_1 10^{K-1} + n_2 10^{K-2} + \cdots + n_{K-1} 10^1 + n_K$.

(e.g. $7392 = 7 \cdot 1000 + 3 \cdot 100 + 9 \cdot 10 + 2 = 7 \cdot 10^3 + 3 \cdot 10^2 + 9 \cdot 10^1 + 2$)

Since $10^i \equiv 0 \bmod 10$ for $1 \le i \le K-1$ we have that

$n \equiv n_K \bmod 10$ (i.e. the remainder of $10 \overline{)n}$ is the last digit n_K).

So, to find the last digit of 4^{100}, it suffices to compute $4^{100} \bmod 10$.

$4^1 \equiv 4 \bmod 10$ $\quad 4^3 \equiv 24 \overset{\bmod 10}{\equiv} 4 \bmod 10$ ... etc.
$4^2 = 16 \equiv 6 \bmod 10$ $\quad 4^4 \equiv 16 \bmod 10 \equiv 6 \bmod 10$

So, $4^K \bmod 10 \equiv 4$ if K odd

$\equiv 6$ if K even.

$\therefore 4^{100} \equiv 6 \bmod 10 \Rightarrow$ last digit of 4^{100} is 6.