Mat 202: Tutorial #6

Recall:

- A <u>relation</u> b/w sets S & T is a subset of $S \times T$. A relation on S is a subset of $S \times S$.

- An <u>equivalence relation</u> on a set S is a relation R on S s.t. $\forall\, x, y, z \in S$:
  - (a) <u>Reflexive</u>: $(x,x) \in R$ $(x \sim x)$
  - (b) <u>Symmetric</u>: $(x,y) \in R \Rightarrow (y,x) \in R$
  - (c) <u>Transitive</u>: $(x,y) \in R$ & $(y,z) \in R \Rightarrow (x,z) \in R$.

- An <u>order relation</u> is a relation that is reflective, transitive, and <u>antisymmetric</u> ($(x,y) \in R$ & $(y,x) \in R \Rightarrow x = y$.

1. Which of the following are equivalence relations?

(a) $\{(a,b) \mid a$ & b are the same age$\}$.

$x \sim x$. ✓
$x \sim y \Rightarrow y \sim x$. ✓
$x \sim y$ & $y \sim z \Rightarrow x \sim z$ ✓ Yes.

(b) $\{(a,b) \mid a$ & b have the same parents$\}$.

$x \sim x$ ✓, $x \sim y \Rightarrow y \sim x$. ✓, $x \sim y$ & $y \sim z \Rightarrow x \sim z$. ✓ Yes.

(c) $\{(a,b) \mid a$ & b share a common parent$\}$.

$x \sim x$. ✓ $x \sim y \Rightarrow y \sim x$ ✓ <u>Not</u> transitive though: x & y could have the same mom, but different dads, y & z could have same dad but different moms. x & z need not share a parent.

$\boxed{d}$ $\{(a,b) \mid a$ & b have met$\}$

Not transitive.

$\boxed{e}$ $\{(a,b) \mid a$ & b speak common language$\}$

Not transitive.

2. Which of the following are equivalence relations or order relations?

$\boxed{a}$ $\{(a,b) \mid a$ is an ancestor of $b\}$. Suppose every individual is an ancestor of itself.

$x \sim x$. ✓ $x \sim y \not\Rightarrow y \sim x$. Not symmetric.
$x \sim y$ & $y \sim x \Rightarrow x = y$ ✓ $x \sim y$ & $y \sim z \Rightarrow x \sim z$. ✓
$\therefore$ order relation.

$\boxed{b}$ $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 y_2 = x_2 y_1$, where x_1, y_1, x_2, y_2 positive integers.

$(x_1, y_1) \sim (x_1, y_1)$, since $x_1 y_1 = y_1 x_1$. ✓

$(x_1, y_1) \sim (x_2, y_2) \Rightarrow x_1 y_2 = x_2 y_1 \Rightarrow (x_2, y_2) \sim (x_1, y_1)$. ✓

Suppose $(x_1, y_1) \sim (x_2, y_2)$ & $(x_2, y_2) \sim (x_3, y_3)$

$\Rightarrow \underbrace{x_1 y_2 = x_2 y_1}$ & $x_2 y_3 = y_2 x_3$. WTS $x_1 y_3 = x_3 y_1$.

$x_2 = \dfrac{x_1 y_2}{y_1}$

So, $\dfrac{x_1 \cancel{y_2}}{y_1} y_3 = \cancel{y_2} x_3 \Rightarrow x_1 y_3 = x_3 y_1 \Rightarrow (x_1, y_1) \sim (x_3, y_3)$. ✓

$\therefore$ Equivalence relation.

3. On the set $\{1,2,3\}$ consider the equivalence relations

(1,2) means $1 \sim 2$.

$R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ &

$R_2 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$.

$\boxed{a}$ Is $R_1 \cup R_2$ an equivalence relation?

(1,2) & (2,3) are in our set, so $1 \sim 2$ & $2 \sim 3$.
But $1 \not\sim 3$ b/c (1,3) not in $R_1 \cup R_2$
$\Rightarrow$ $R_1 \cup R_2$ not have transitive property $\Rightarrow$ not an equivalence relation.

Recall: Given a set S and an equivalence relation on S, the set of elements equivalent to $x \in S$ is the equivalence class containing x.

$\boxed{b}$ List the equivalence classes of R_1. List the equivalence classes of R_2.

R_1: $[1] = \{1, 2\}$, $[3] = \{3\}$.

R_2: $[1] = \{1\}$. $[2] = \{2,3\}$.

4. The relation R on $\mathbb{Z}$ is defined by $x \sim y$ if $x+3y$ is even. Prove that this is an equivalence relation & find the equivalence classes.

- $x \sim x$ since $x+3x = 4x = 2(2x)$ is even. ✓
- $x \sim y \Rightarrow \underbrace{x+3y}_{=:2k}$ even $\Rightarrow$ $x = 2k-3y$. Then $y+3x = y+3(2k-3y)$

$= y+6k-9y = 6k-8y = 2(3k-4y)$ even $\Rightarrow$ $y \sim x$. ✓

- Suppose $x \sim y$ and $y \sim z \Rightarrow x+3y = 2k$ & $y+3z = 2l$
$\Rightarrow x = 2k-3y$ & $3z = 2l-y$
$\Rightarrow x+3z = 2k-3y+2l-y = 2k+2l-4y = 2(k+l-2y)$ even
$\Rightarrow x \sim z$. ✓

$\therefore$ Equivalence relation.

$[0] = \{x \in \mathbb{Z} \mid x \sim 0\} = \{x \in \mathbb{Z} \mid x+3(0) \text{ even}\} = \{x \in \mathbb{Z} \mid x \text{ even}\}$.

$[1] = \{x \in \mathbb{Z} \mid x \sim 1\} = \{x \in \mathbb{Z} \mid x+3(1) \text{ even}\} = \{x \in \mathbb{Z} \mid x \text{ odd}\}$.

odd + odd = even
odd + even = odd

This partitions $\mathbb{Z}$ into two distinct equivalence classes.

- Recall: Given $n \in \mathbb{N}$, $x^{\in \mathbb{Z}}$ & $y^{\in \mathbb{Z}}$ are congruent modulo n if $x-y$ is divisible by n. i.e. $x \equiv y \mod n$. This gives an equivalence relation on $\mathbb{Z}$.