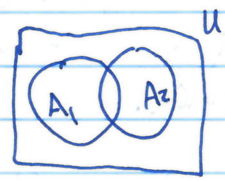
Mat 202 - Tutorial #4

Inclusion - Exclusion Principle: Given a universe U of items and subsets $A_1, \dots, A_n$ of the items, the number N_ϕ of items belonging to none of the subsets is given by:

[when $S = \emptyset$, the count is $|U|$, b/c each element is in every one of no sets].

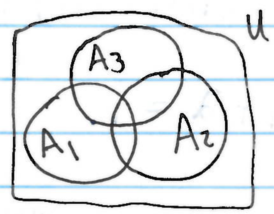
$$N_\phi = \sum_{S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|.$$

e.g., $n=2$:



$$N_\phi = |U| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

$n=3$:



$$N_\phi = |U| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|) - |A_1 \cap A_2 \cap A_3|.$$

1. (a) How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?

A_1 = multiples of 3 b/w 1 & 1000. $|A_1| = \left\lfloor \frac{1000}{3} \right\rfloor = 333$.
A_2 = multiples of 5 b/w 1 & 1000. $|A_2| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$.

$A_1 \cap A_2$ = multiples of 3 & 5 (i.e., multiples of 15):

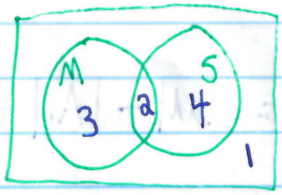
$$|A_1 \cap A_2| = \left\lfloor \frac{1000}{15} \right\rfloor = 66.$$

So, $|A_1| + |A_2| - |A_1 \cap A_2| = 333 + 200 - 66 = 467$ integers b/w 1 & 1000 are either multiples of 3 or 5.

$\boxed{b}$ How many do <u>not</u> have a divisor in the set $\{3, 5\}$?

$1000 - (|A_1| + |A_2|) + (|A_1 \cap A_2|) = 1000 - 467 = 533.$

2. Among 10 students, 5 study math, 6 study science, and 2 study both. How many students study neither math nor science?



From the picture, we can see that 1 person studies neither math nor science.

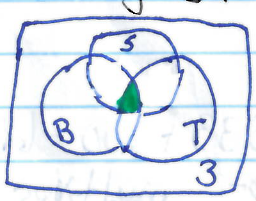
By Inclusion-Exclusion: $|U| - (|M| + |S|) + (|M \cap S|)$

$= 10 - (5 + 6) + 2 = 10 - 11 + 2 = 1.$

3. Given a class of 50 students, suppose you know the following:

30 play baseball
18 play tennis
26 play soccer
9 play baseball & tennis
16 play baseball & soccer
8 play tennis & soccer
47 play at least one of these sports.

How many students play all 3 sports?



$3 = |U| - (|B| + |T| + |S|)$

$+ (|B \cap T| + |B \cap S| + |T \cap S|) - |B \cap T \cap S|$

$\Rightarrow 3 = 50 - (30 + 18 + 26) + (9 + 16 + 8) - |B \cap T \cap S|$

$\Rightarrow 3 = 50 - 74 + 33 - |B \cap T \cap S|$

$\Rightarrow |B \cap T \cap S| = 6.$ $\therefore$ 6 students play all 3 sports.

$\begin{array}{r} 3 \\ 15 \\ 7 \\ \hline 105 \end{array}$

4. How many positive integers between 1 & 1000 are relatively prime to 105?

$105 = 3 \cdot 5 \cdot 7.$

$A_1 =$ multiples of 3. $|A_1| = \left\lfloor \frac{1000}{3} \right\rfloor = 333.$

$A_2 =$ multiples of 5. $|A_2| = \left\lfloor \frac{1000}{5} \right\rfloor = 200.$

$A_3 =$ multiples of 7. $|A_3| = \left\lfloor \frac{1000}{7} \right\rfloor = 142.$

$A_1 \cap A_2$: multiples of 3 & 5 (i.e. 15): $|A_1 \cap A_2| = \left\lfloor \frac{1000}{15} \right\rfloor = 66.$

$A_1 \cap A_3$: multiples of 3 & 7 (i.e. 21): $|A_1 \cap A_3| = \left\lfloor \frac{1000}{21} \right\rfloor = 47.$

$A_2 \cap A_3$: multiples of 5 & 7 (i.e. 35): $|A_2 \cap A_3| = \left\lfloor \frac{1000}{35} \right\rfloor = 28.$

$A_1 \cap A_2 \cap A_3$: multiples of 3, 5, & 7 (i.e. 105): $|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{1000}{105} \right\rfloor = 9.$

$1000 - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$

$= 1000 - 333 - 200 - 142 + 66 + 47 + 28 - 9 = \boxed{457}.$

5. How many decimal n-tuples contain at least one of each of $\{1,2,3\}$? [i.e.] i.e.] We want it to have at least one 1, at least one 2, & at least one 3].

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \cdots\ \underset{n}{\_}$$

If $n=6$, for example,
We want things like 514321 but not 111222 [b/c no 3]

A_1: n-tuples w/ no 1's : $|A_1| = 9^n$
A_2: n-tuples w/ no 2's : $|A_2| = 9^n$
A_3: n-tuples w/ no 3's : $|A_3| = 9^n$.

$A_1 \wedge A_2$: n-tuples w/ no 1's & 2's : $|A_1 \wedge A_2| = 8^n$
$A_1 \wedge A_3$: n-tuples w/ no 1's & 3's : $|A_1 \wedge A_3| = 8^n$
$A_2 \wedge A_3$: n-tuples w/ no 2's & 3's : $|A_2 \wedge A_3| = 8^n$

$A_1 \wedge A_2 \wedge A_3$: n-tuples w/ no 1's & 2's & 3's : $|A_1 \wedge A_2 \wedge A_3| = 7^n$.

Total # of n-tuples: 10^n.

$\therefore$ There are

$10^n - 3\cdot 9^n + 3\cdot 8^n - 7^n$ many.

$\underset{2\cdot 3}{6}$ $\underset{2\cdot 5}{10}$ $\underset{3\cdot 5}{15}$

6. How many natural numbers less than 20,000 have no divisor in $\{6, 10, 15\}$?

LCM(6,10) = 30.
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A_1: mult. of 6 : $\left\lfloor \frac{19{,}999}{6} \right\rfloor = 3333$.
A_2: mult. of 10 : $\left\lfloor \frac{19999}{10} \right\rfloor = 1999$.
A_3: mult of 15 : $\left\lfloor \frac{19999}{15} \right\rfloor = 1333$.

$A_1 \wedge A_2$: mult. 30 : $\left\lfloor \frac{19999}{30} \right\rfloor = 666$.
$A_1 \wedge A_3$: mult. 30 : $\left\lfloor \frac{19999}{30} \right\rfloor = 666$.
$A_2 \wedge A_3$: mult. 30 : $\left\lfloor \frac{19999}{30} \right\rfloor = 666$.
$A_1 \wedge A_2 \wedge A_3$: mult. 30 : $\left\lfloor \frac{19999}{30} \right\rfloor = 666$.

$19999 - 3333 - 1999 - 1333 + 3\cdot 666 - 666 = \boxed{14{,}666}$.