Mat 202 - Tutorial #3

Pigeonhole Principle: Placing more than kn objects into n classes puts more than k objects into some class.

i.e.) Given a set of m objects and n groups to place them in with $n < m$, then one group must have $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ objects in it.

round down to nearest integer

1. Given 14 people, each can receive a grade of A, B, or C.

[a] How many people must share a grade?

Here $m = 14$, $n = 3$.

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{13}{3} \right\rfloor + 1 = 4 + 1 = 5.$$

At least 5 people must share a grade.

[b] How many people would be needed in order for the Pigeonhole Principle to guarantee that 9 people share a grade?

$$\left\lfloor \frac{m-1}{3} \right\rfloor + 1 = 9 \iff \left\lfloor \frac{m-1}{3} \right\rfloor = 8 \Rightarrow m-1 = 24, 25, \text{ or } 26$$

$\Rightarrow m = 25, 26,$ or 27.

$\therefore$ At least 25 people would be needed.

$\boxed{c}$ How many people to ensure 2 People get an A?

No number suffices since the Pigeonhole Principle cannot Force pigeons into a specific hole.

2. Suppose you are given 27 different numbers between 15 and 60 (inclusive). Show that some pair differ by exactly 6.

Let's denote our 27 numbers by $x_1, x_2, \ldots, x_{27}$.

We want to show that there exists an $1 \le i, j \le 27$ such that $x_j - x_i = 6 \iff x_j = x_i + 6$.

Consider the two sets

$$15 \le x_1 < x_2 < \cdots < x_{27} \le 60$$

$$21 \le x_1 + 6 < x_2 + 6 < \cdots < x_{27} + 6 \le 66.$$

We wish to show that these two sets have an element in common.

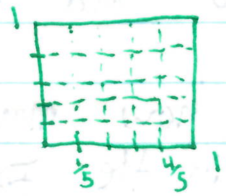
There are at most $66 - 15 + 1 = 52$ distinct integers in these 2 sets. But there are $27 + 27 = 54$ elements in the 2 sets.

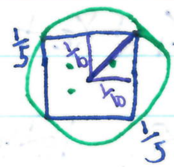
$\left\lfloor \frac{54-1}{52} \right\rfloor + 1 = 1 + 1 = 2$. By Pigeonhole Principle,

at least 2 elements in these sets must be the same integer $\Rightarrow \exists\, i \neq j$ s.t. $x_j = x_i + 6 \iff x_j - x_i = 6$. ■

3. [#20] Suppose 51 points are chosen at random inside a 1×1 square. Prove that 3 of these points are inside of a circle of radius $1/7$.

Subdivide box into 25 sub boxes of length $1/5$.





$$\left\lceil \frac{51-1}{25} \right\rceil + 1 = 2+1 = 3.$$

There must exist at least one subbox which contains 3 points.

The circle circumvented around the square has radius $\sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^2} = \sqrt{2/100} = \sqrt{1/50} < \sqrt{1/49} = 1/7$.

So, this circle sits inside of a circle of radius $1/7$ $\Rightarrow$ the 3 points lie in a circle of radius $1/7$.

4. [#3] How many nonempty collections of letters can be formed from 4 A's and 8 B's?

Want (A, B) where have $0 \le A \le 4$, $0 \le B \le 8$, and want to exclude $(0,0)$. i.e. $(2,3) \leftrightarrow AABBB$.
5 choices for A and 9 choices for B:
$5 \cdot 9 - 1 = 45 - 1 = 44$ nonempty collections.