- ms.math.mcmaster.ca/lauren.dedeu
- office Hours: Wed. 2:30-3:30PM DH2027

Mat 202 - Tutorial #1

1. Count the number of n-digit binary sequences with exactly r 1's (i.e., $n-r$ 0's).

order doesn't matter!

Recall: The number of r-subsets of an n-set is $\binom{n}{r} = \frac{n!}{K!(n-K)!}$.

We can think about having n slots:

$$\underset{1}{\_\_}\ \underset{2}{\_\_}\ \underset{3}{\_\_}\ \cdots\ \underset{n-1}{\_\_}\ \underset{n}{\_\_}$$

We need r slots to be 1 & $n-r$ slots to be 0.

How many ways are there to choose r slots?

↳ There are $\binom{n}{r}$ ways.

$\therefore$ There are $\binom{n}{r}$ different ways we can place the 1's in r slots $\Rightarrow$ there are $\binom{n}{r}$ many such binary sequences.

2. a) How many ways are there to seat 6 different boys & 6 different girls along one side of a long table with 12 seats?

b) How many ways if boys & girls alternate seats?

order matters!

Recall: The number of arrangements of K distinct elements from a set of size n is:

$$P(K,n) = n(n-1)\cdots(n-K+1) = \frac{n!}{(n-K)!}.$$

a) We have 12 people and we want to seat them in 12 seats:

$$\underset{1}{12}\cdot\underset{2}{11}\cdot\underset{3}{10}\cdot\underset{4}{9}\cdot\underset{5}{8}\cdot\underset{6}{7}\cdot\underset{7}{6}\cdot\underset{8}{5}\cdot\underset{9}{4}\cdot\underset{10}{3}\cdot\underset{11}{2}\cdot\underset{12}{1} = 12!$$

12! ways!

i.e. $P(12,12) = \frac{12!}{(12-12)!} = \frac{12!}{1} = \underbrace{12!}_{479,001,600}.$

b) Either all boys sit in odd seats or all boys sit in even seats.

There are $P(6,6) = 6!$ ways to sit the boys in the 6 odd seats. There are $P(6,6) = 6!$ ways to sit the girls in the 6 even seats.

$\therefore$ There are $\underbrace{2\cdot 6!\cdot 6!}_{1,036,800}$ ways.

3. (a) How many arrangements are there of the 7 letters in the word "UNUSUAL"?

First notice that there are 3 U's, & 1 of each other letter.

- How many ways are there to position the U's?

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$
1 2 3 4 5 6 7

[Same letter so order doesn't matter...]

There are $\binom{7}{3} = \frac{7!}{4!3!}$ ways.

- Once we position the U's, how many ways can we arrange the other letters?

Once the U's are placed there are 4 spots left.

$\therefore P(4,4) = \frac{4!}{0!} = 4!$ ways.

$\therefore$ There are $\left(\frac{7!}{4!3!}\right)(4!) = \underbrace{\frac{7!}{3!}}_{840}$ ways.

<u>Note</u>: You could also think about this in the following way:

7 letters can be arranged 7! ways. Since there are 3 U's, you can rearrange them in the word 3! ways without changing the word. $\therefore$ It's $\frac{7!}{3!}$.

$\boxed{6}$ How many ways can you arrange the letters in the word MISSISSIPPI.

1 M
4 I's
4 S's
2 P's

$\binom{11}{4}$ ways to arrange the I's.

$\binom{7}{4}$ ways to arrange the S's, once the I's have been placed.

$\binom{3}{2}$ ways to arrange the P's, once the I's & S's have been placed.

$\binom{1}{1}$ ways to arrange M, once the other letters have been placed.

$$\therefore \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1} = \frac{11!}{7!\,4!} \cdot \frac{7!}{4!\,3!} \cdot \frac{3!}{2!\,1!} \cdot \frac{1!}{1!\,0!} = \underbrace{\frac{11!}{4!\,4!\,2!}}_{34,650} \text{ ways.}$$

4. What is the probability that a 5-card poker hand has the following?

[a] Four Aces

Recall: The probability that an event will occur is:

$$\frac{\text{\# of ways an event can occur}}{\text{\# of possible outcomes}}.$$

Here, a deck has 52 cards, therefore there are $\binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960$ $\underbrace{\text{possible poker hands}}_{\text{possible outcomes}}$.

- How many ways can we choose 4 Aces?
  $\binom{4}{4} = 1$ way.

- Once we have 4 Aces, how many options do we have for the 5th card?
  ↳ 48

∴ The are 48 different poker hands with 4 aces.

∴ The probability is $\frac{48}{\binom{52}{5}} = \frac{1}{54{,}145}$.

[b] Four of a Kind.

By [a] there are 48 different poker hands with 4 cards of the same suit. There are 13 different ranks (types of cards).
∴ There are $13 * 48 = 624$ such hands ⇒ the probability is $\frac{624}{\binom{52}{5}} = \frac{1}{4165}$.

$\boxed{c}$ A Full house. [i.e.] 3 of one suit & 2 of a different suit].

- How many ways to choose 3 of one rank?

$\binom{13}{1}$ ways to choose one rank.

$\binom{4}{3}$ ways to choose 3 cards from the rank we chose.

$\therefore \binom{13}{1}\binom{4}{3}$ ways.

- How many ways to choose 2 of a different rank?

$\binom{12}{1}$ ways to choose one rank.

$\binom{4}{2}$ ways to choose 2 cards from the rank we chose.

$\therefore \binom{12}{1}\binom{4}{2}$ ways.

$\therefore$ There are $\underbrace{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}_{\shortparallel}$ different full house poker hands.

$$13 \cdot \frac{4!}{3!} \cdot 12 \cdot \frac{4!}{2!2!} = 3744$$

$\therefore$ The probability is $\dfrac{3744}{\binom{52}{5}} = \dfrac{6}{4165}$.

5. In a race with 10 horses, the 1^{st}, 2^{nd}, & 3^{rd} place finishers are noted. How many outcomes are there?

$$P(k,n) = \frac{n!}{(n-k)!}$$

$$P(3,10) = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720.$$

# of arrangements of 3 distinct elements from a group of 10

6. A bridge hand consists of 13 cards.

(a) How many bridge hands are there?

$$\binom{52}{13} = 635{,}013{,}559{,}600.$$

(b) What is the probability that a bridge hand has 4 spades (& 9 cards that are not spades)?

13 spades. $\binom{13}{4}$ is the number of subsets of spades of size 4.

39 non-spades. $\binom{39}{9}$ subsets of non-spades of size 9.

$\therefore$ $\binom{13}{4}\binom{39}{9}$ possible hands.

Probability is:
$$\frac{\binom{13}{4}\binom{39}{9}}{\binom{52}{13}} \approx 0.2386.$$

$\boxed{c}$ What is the probability that a bridge hand has 4 spades & 3 hearts (i.e.? remaining cards are diamonds & clubs).

$$\frac{\overbrace{\binom{13}{4}}^{\text{spades}}\overbrace{\binom{13}{3}}^{\text{hearts}}\overbrace{\binom{26}{6}}^{\text{diamonds/clubs}}}{\binom{52}{13}} \approx 0.0741.$$