

λ -Harmonious Graph Colouring

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ABSTRACT

In 1983, Hopcroft and Krishnamoorthy defined a new type of graph colouring called harmonious colouring. Harmonious colouring is a proper vertex colouring such that no two edges share the same colour pair. The least number of colours needed to harmoniously colour a graph is called the harmonious chromatic number. We will examine the results found for the harmonious chromatic number of paths, cycles, and trees. We will also extend the definition of harmonious colouring and define λ -harmonious colouring, which allows each edge colour pair to occur up to λ times. We will explore λ -harmonious colouring and will prove some results for the λ -harmonious chromatic number of complete graphs, complete bipartite graphs, paths, cycles, and wheels.

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1. INTRODUCTION

Graph colouring originated in the mid-nineteenth century, when mathematicians began to ask the question, “Can a geographical map always be coloured using 4 colours or less?” Although this problem took almost a century to solve, the minimum number of colours needed to properly colour different types of graphs has been studied extensively in the literature [1]. The two main types of graph colouring are vertex colouring and edge colouring. In this paper, we will explore vertex colouring.

Vertex colouring is an assignment of colours to the vertices of a graph such that each vertex receives exactly one colour. A colouring is called proper if no two adjacent vertices share the same colour. In 1983, Hopcroft and Krishnamoorthy defined a new type of vertex colouring called harmonious colouring [3]. Harmonious colouring is a special type of graph colouring in which each edge is assigned a distinct colour pair, i.e. if one edge has the colours red and blue on its incident vertices, then no other edge can also have the colour pair {red,blue}. In this report, we will discuss several well-known harmonious colouring results for families of simple graphs, and will extend these results to allow for each colour pair to occur up to λ times.

In Section 3, we will discuss harmonious colouring. Here we will state the theorems for paths, cycles, and trees, and will outline the proofs for paths and cycles. In Section 4, we will extend the harmonious colouring results and discuss λ -harmonious colouring, which allows each edge colour pair to occur up to λ times. Here we will give the λ -harmonious chromatic number for complete graphs, complete bipartite graphs, paths, cycles, and wheels.

2. TERMINOLOGY

For additional terminology, see [2]. We will denote

path - an alternating sequence of distinct vertices and edges that begins and ends with a vertex.

For example, $P : v_1v_2v_3$ would be a path of length 2, denoted P_3 .

cycle - a closed path where the first and last vertex are the same. For example $H = v_1v_2v_3v_1$ would be a cycle of length 3, denoted C_3 (also known as a 3-cycle or a triangle).

tree - a connected graph without cycles.

$|V(G)|$ - the number of vertices in a graph G .

$|E(G)|$ - the number of edges in a graph G .

degree - the degree of a vertex v , or $\deg(v)$, refers to the number edges incident to v .

vertex colouring - an assignment of colours to the vertices of a graph, such that each vertex receives exactly one colour.

proper colouring - a vertex colouring such that no two adjacent vertices share the same colour.

edge colour pair - the pair of colours assigned to an edge's incident vertices.

harmonious colouring - a proper colouring such that no two edges share the same colour pair.

For example, if an edge's vertices are coloured red and blue, then there is no other edge with the colour pair $\{red, blue\}$.

$h(G)$ - the least number of colours needed to harmoniously colour a graph G ; called the harmonious chromatic number.

λ -harmonious colouring - a proper colouring such that no $\lambda+1$ edges share the same colour pair.

Note that the traditional harmonious colouring defined above is a 1-harmonious colouring.

$h_\lambda(G)$ - the least number of colours needed to λ -harmoniously colour a graph G ; called the λ -harmonious chromatic number.

complete graph - a graph in which each pair of vertices is connected by an edge; we denote a complete graph on n vertices by K_n .

eulerian path - a trail in a graph that visits each edge exactly once.

eulerian cycle - a closed trail in a graph that visits each edge exactly once.

complete bipartite graph - a graph whose vertex set can be decomposed into two disjoint sets such that no two vertices in the same set are adjacent, and every pair of vertices in distinct sets are adjacent; we will denote a complete bipartite graph by $K_{m,n}$ where m and n are the sizes of the disjoint sets, with $m \geq n$.

star - a complete bipartite graph such that $n = 1$.

wheel - a graph on n vertices formed by connecting a single vertex to all vertices of a C_{n-1} ; we denote a wheel on n vertices as W_n .

3. HARMONIOUS GRAPH COLOURING

The harmonious chromatic number, $h(G)$, is the minimum number of colours needed to properly colour a graph G such that no two edges share the same colour pair. This parameter was originally introduced by Hopcroft and Krishnamoorthy in 1983 [3], and since then, $h(G)$ has been found for several different families of graphs.

We can gain a lower bound for $h(G)$ by considering the requirement that each edge receives a unique colour pair. The binomial coefficient $\binom{k}{2}$ tells us how many ways k colours can be arranged into pairs of 2. Therefore, we know that $\binom{k}{2} \geq |E(G)|$. Let k be the smallest integer such that this inequality holds. This gives us a lower bound for the harmonious chromatic number: $h(G) \geq k$.

Note that our definition of the harmonious chromatic number differs slightly from how it was first introduced. Hopcroft and Krishnamoorthy's definition of $h(G)$ did not require that G was properly coloured. The majority of the literature, however, has since added the additional condition that G is properly coloured, so we will use this variation of $h(G)$ as well.

We will now briefly state the results found for the harmonious chromatic number of paths, trees, and cycles. We will also outline the proofs for P_n and C_n , as Section 4 will use these proof techniques to find the λ -harmonious chromatic number of paths and cycles.

Theorem 1: [6] The harmonious chromatic number of a path with n vertices is as follows:

Let $r \in \mathbb{Z}$ be determined by the inequality $\binom{2r-1}{2} < n - 1 \leq \binom{2r+1}{2}$. Then

$$h(P_n) = \begin{cases} 2r & \text{if } n - 1 \leq \binom{2r}{2} - (r - 1) , \\ 2r + 1 & \text{otherwise.} \end{cases}$$

Proof: We can determine $h(P_n)$ by considering smallest m such that the complete graph on m vertices, K_m , has a subgraph with an eulerian path of length $n - 1$. An eulerian path of length $n - 1$ corresponds to a harmonious colouring of P_n , since if we colour the m vertices of K_m with m different colours, this means that our eulerian path of length $n - 1$ is properly coloured such that each edge has a distinct colour pair. Since we are considering the smallest m such that this is true, we know that the harmonious chromatic number of P_n will be m .

We know that a graph has an eulerian path if and only if it is connected and has at most two vertices of odd degree [2]. Also note that complete graphs K_m have $\binom{m}{2}$ edges, so when we consider

complete graphs, we know that our criterion $\binom{k}{2} \geq |E(G)|$ is satisfied for $k = m$. If m is odd, then all of the vertices of K_m have even degree, so $n - 1 \leq \binom{m}{2}$, i.e. the length of P_m must be less than or equal to the number of edges in K_m . If m is even, then all vertices of K_m have odd degree, so we must delete at least $\frac{m}{2} - 1$ edges to ensure we have at most 2 vertices of odd degree. Therefore, if m is even, then we must have $n - 1 \leq \binom{m}{2} - (\frac{m}{2} - 1)$. The theorem then follows, letting $r = \frac{m}{2}$.

For example, in Figure 3.1 we can see that K_6 has 15 edges, and each vertex has odd degree. If we remove 2 edges, v and w , we are left with only 2 vertices of odd degree. Therefore, we can trace an eulerian path 13524614512632, and can see that $h(P_{11}) = h(P_{12}) = h(P_{13}) = h(P_{14}) = 6$.

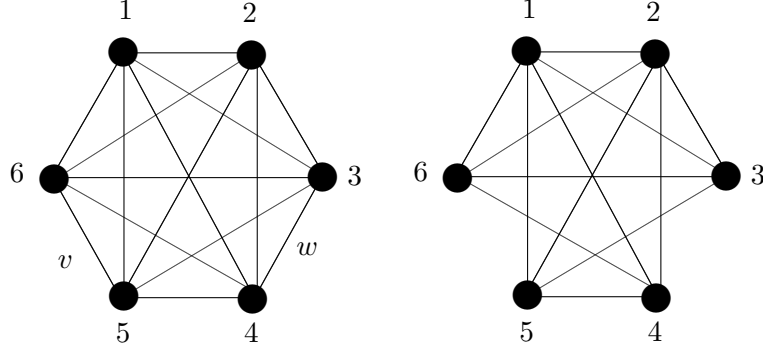


Fig. 3.1: $K_6 - \{v, w\}$ has an eulerian path of length 13.

The harmonious chromatic number of a path can also be stated as follows [7]:

Let k be the least integer such that $n - 1 \leq \binom{k}{2}$. Then,

$$h(P_n) = \begin{cases} k & \text{if } k \text{ odd,} \\ & \text{or, if } k \text{ even and } n - 1 = \binom{k}{2} - i, \text{ for } i \in \{\frac{k}{2} - 1, \frac{k}{2}, \dots, k - 2\}, \\ k + 1 & \text{otherwise.} \end{cases}$$

Since the harmonious chromatic number of a path is either k or $k + 1$, one might expect that the harmonious chromatic number of a tree would also be close to k . However, this is not the case. John Mitchem [7] proved that the harmonious chromatic number of a tree T with n vertices can fall anywhere between k and n . We will state the theorem formally:

Theorem 2: [7] Let k be the least integer such that $n - 1 \leq \binom{k}{2}$. Then for each t such that $k \leq t \leq n$, there is a tree T with n vertices such that $h(T) = t$.

Theorem 3: [4] [5] The harmonious chromatic number of a cycle with n vertices is as follows: Let k be the least integer such that $n \leq \binom{k}{2}$. Then,

$$h(C_n) = \begin{cases} k & \text{if } k \text{ odd and } n \neq \binom{k}{2} - i, \text{ for } i \in \{1, 2\}, \\ & \text{or, if } k \text{ even and } n \neq \binom{k}{2} - i, \text{ for } i \in \{0, 1, \dots, \frac{k}{2} - 1\}, \\ k + 1 & \text{otherwise.} \end{cases}$$

Proof: Let k be the least integer such that $n \leq \binom{k}{2}$. Then, as with paths, we can determine the harmonious chromatic number of a cycle, C_n , by considering the smallest k such that the complete graph on k vertices, K_k , has a subgraph with an eulerian cycle of length n . We know that a graph has an eulerian cycle if and only if it's connected and each of its vertices has even degree [2].

If k is odd, then each vertex in K_k has even degree. Therefore, in order to create a subgraph J of K_k with an eulerian cycle of length n (i.e. a subgraph such that each vertex has even degree), we need to either delete zero edges, or delete the edges from a cycle in K_k . Removing the edge from a cycle will keep the degree of each vertex even, since vertices in a cycle all have degree 2. We are able to do this unless $n = \binom{k}{2} - i$ for $i = 1$ or 2 , since no cycles of length 1 or 2 exist. So, if k is odd and $n \neq \binom{k}{2} - i$, for $i \in \{1, 2\}$, then $h(C_n) = k$.

For example, if $n = 7$, then $k = 5$. So, if we delete the edges from a cycle of length 3 in K_5 , then we are left with a subgraph of K_5 with an eulerian cycle of length 7 (see Figure 3.2). Therefore, $h(C_7) = k = 5$.

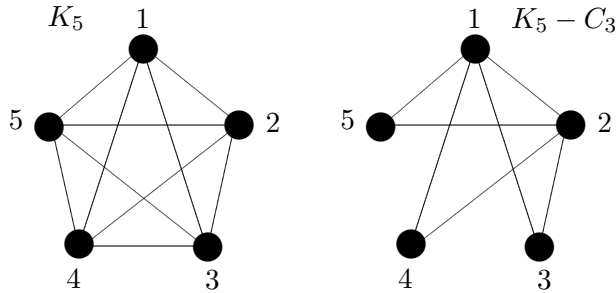


Fig. 3.2: An eulerian subgraph of K_5 with 7 edges.

If $n = \binom{k}{2} - i$ for $i = 1$ or 2 , then we know that K_k has no eulerian subgraph of length n . However, we will show that such a subgraph does exist in K_{k+1} . Let's call this subgraph L . If $i = 1$, then we can create L by tracing a path of length 4 in K_k , deleting the edges from P_4 , and joining the end vertices of P_4 to a new vertex v not in K_k . This new graph L has $\binom{k}{2} - 1$ edges, $k + 1$ vertices, and all edges have even degree. Therefore, L is a subgraph of K_{k+1} and has a eulerian cycle of length $\binom{k}{2} - 1$. So, $h(C_n) = k + 1$ when $n = \binom{k}{2} - 1$. Similarly, if $i = 2$, then we can create L by deleting the edges from a P_5 in K_k , and joining the end vertices of this path to a new vertex v . Therefore, we also have $h(C_n) = k + 1$ for $n = \binom{k}{2} - 2$.

For example, in Figure 3.3 we can see that if $n = 9$, then we can delete the edges from a P_4 in K_5 and join the end vertices of this path to v . If $n = 8$, we delete the edges from a P_5 in K_5 , and join the end vertices to v .

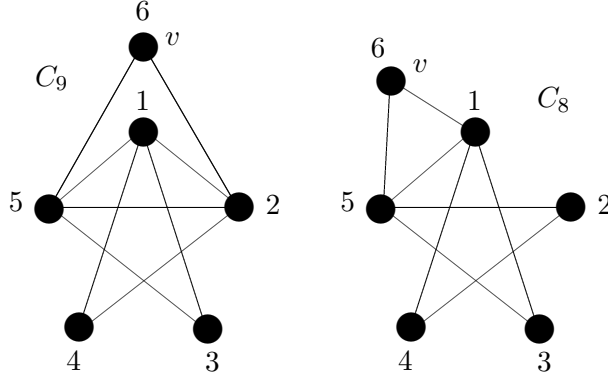


Fig. 3.3: Eulerian subgraphs of K_6 with 9 and 8 edges.

When k is even, we know that all vertices in K_k have odd degree. Therefore, we must delete at least $\frac{k}{2}$ edges from K_k in order to create a subgraph of K_k such that all vertices have even degree. So, let's consider $n = \binom{k}{2} - i$ for $i = \frac{k}{2}, \dots, k - 2$. Note that we only consider i up to $k - 2$, because $\binom{k}{2} - (k - 1) = \binom{k-1}{2}$, and so if $i = k - 1$, then k would not be the minimum integer that satisfies $n \leq \binom{k}{2}$. Let $i = \frac{k}{2} + t$. We can create an eulerian subgraph of K_k with $\binom{k}{2} - i$ edges as follows. If $t = 0$, then remove $\frac{k}{2}$ non-adjacent vertices from K_k . This will make the degree all k vertices even. If $t \neq 0$, then remove from K_k the edges of a star $K_{2t+1,1}$ as well as $\frac{k}{2} - t - 1$ edges that are not adjacent to each other, or to any of the edges of $K_{2t+1,1}$. Doing so will remove $2t + 1 + \frac{k}{2} - t - 1 = \frac{k}{2} + t = i$ edges from K_k , and will make the degree all k vertices even. Therefore, $h(C_n) = k$ for $n = \binom{k}{2} - i$ when $i = \frac{k}{2}, \dots, k - 2$, since there exists an eulerian subgraph of K_k of size n .

For example, when $n = 11$, we have $k = 6$, $i = 4$ and $t = 1$. Therefore, we can create our eulerian subgraph by deleting the edges from a $K_{3,1}$ in K_6 and deleting one other edge which is not adjacent to any of the edges of $K_{3,1}$ (see Figure 3.4).

When $i < \frac{k}{2}$, it's not possible to create an eulerian subgraph of K_k , but we can create an eulerian subgraph of K_{k+1} . Since k is even, we know that $k + 1$ is odd, so all of K_{k+1} 's vertices have even degree. Let $t = \binom{k+1}{2} - n$. If $t \leq k + 1$, then we can create an eulerian subgraph of K_{k+1} , by deleting the edges from a cycle of length t in K_{k+1} . If $t > k + 1$, then we will not be able to trace a cycle of length t in K_{k+1} without repeating an edge. Therefore, we will instead create our eulerian subgraph of size n by deleting t edges from two edge disjoint cycles in K_{k+1} , such that the two cycles use all $k + 1$ vertices. Therefore, $h(C_n) = k + 1$ for $n = \binom{k}{2} - i$ when $i = 0, \dots, \frac{k}{2} - 1$.

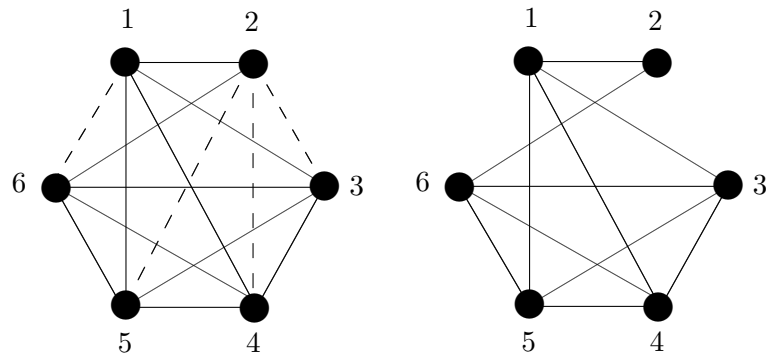


Fig. 3.4: An eulerian subgraph of K_6 with 11 edges.

For example, when $n = 13$, we have $k = 6$ and $t = 8$. So, we can create our eulerian subgraph of K_7 by deleting the edges from two C_4 's in K_7 (see Figure 3.5).

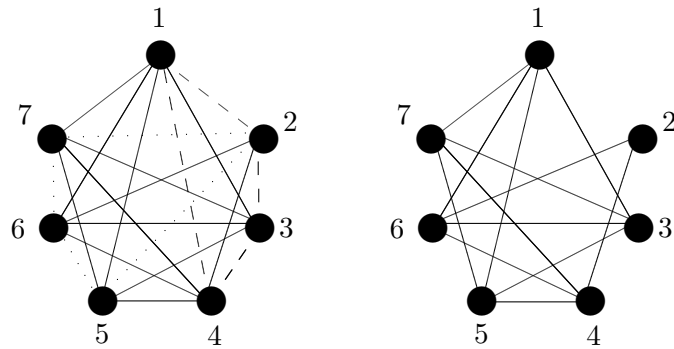


Fig. 3.5: An eulerian subgraph of K_7 with 13 edges.

4. λ -HARMONIOUS GRAPH COLOURING

In the previous chapter we explored the harmonious chromatic number: the least number of colours needed to properly colour a graph such that no two edges share the same colour pair. We will now extend these results to allow for a proper colouring such that no $\lambda + 1$ edges may share the same colour pair. We call this colouring λ -harmonious colouring. The least number of colours needed to λ -harmoniously colour a graph G is called the λ -harmonious chromatic number, and will be denoted by $h_\lambda(G)$.

As with harmonious colouring, we can gain a lower bound for the λ -harmonious chromatic number by considering the requirement that no $\lambda + 1$ edges may share the same colour pair. The binomial coefficient $\binom{k}{2}$ tells us how many ways k colours can be arranged into pairs of 2. Therefore, since we are only allowed to use each colour pair at most λ times, we must have $|E(G)| \leq \lambda \binom{k}{2}$. Let k be the smallest integer such that this inequality holds. Then, $h_\lambda(G) \geq k$.

We will now give our results for the λ -harmonious chromatic number of complete graphs, complete bipartite graphs, paths, cycles, and wheels.

Theorem 4: The λ -harmonious chromatic number of a complete graph on n vertices equals n , i.e. $h_\lambda(K_n) = n$.

Proof: By definition, a λ -harmonious colouring must be a proper colouring. Since each vertex in K_n is adjacent to all $n - 1$ remaining vertices, we need n colours in order to properly colour K_n . Therefore, $h_\lambda(K_n) = n$.

Theorem 5: The λ -harmonious chromatic number of a complete bipartite graph $K_{m,n}$ with $m \geq n$ is:

$$h_\lambda(K_{m,n}) = \left\{ \min \left(\left\lceil \frac{m}{\lfloor \frac{\lambda}{q} \rfloor} \right\rceil + \left\lceil \frac{n}{q} \right\rceil ; 1 \leq q \leq \lfloor \sqrt{\lambda} \rfloor, q \in \mathbb{Z} \right) \right\}.$$

Proof:

The proof is quite straightforward for $\lambda = 1, 2$, and 3 , so we will explore these cases before proving the theorem in general. We will denote the disjoint vertex set of size m in $K_{m,n}$ by A , and will denote the the disjoint vertex set of size n by B (see Figure 4.1).

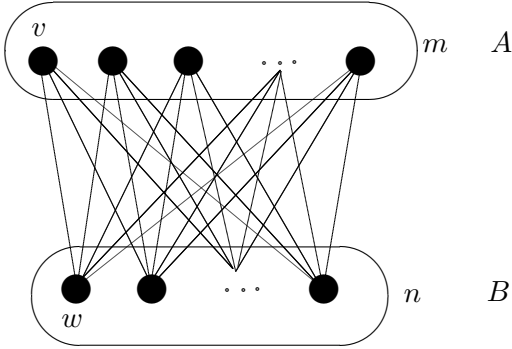


Fig. 4.1: Complete bipartite graph, $K_{m,n}$, with $m \geq n$.

For $\lambda = 1$ (i.e. regular harmonious colouring), we know that no two edges can share the same colour pair. If we give a vertex v in A the colour 1, then all vertices in B must receive distinct colours, since v is adjacent to each vertex in B . Let's suppose w is a vertex in B that received the colour 2. Since w is adjacent to every vertex in A , we must give the vertices in A distinct colours as well. Therefore, $h_1(K_{m,n}) = m + n$.

For $\lambda = 2$, we are allowed to use each colour pair twice. Therefore, the most efficient way to colour $K_{m,n}$ would be to group the vertices of A into groups of 2, giving each group a distinct colour. This forces us to use n distinct colours in B . Since $m \geq n$, it will always be better to repeat colours in A , rather than B . Therefore, $h_2(K_{m,n}) = \lceil \frac{m}{2} \rceil + n$. Similarly, $h_3(K_{m,n}) = \lceil \frac{m}{3} \rceil + n$.

When $\lambda = 4$, things get a little more interesting. Depending on the graph, it may be better to use $\lceil \frac{m}{4} \rceil$ colours in A , and use n colours in B , or sometimes it may be better to use $\lceil \frac{m}{2} \rceil$ colours in A , and $\lceil \frac{n}{2} \rceil$ colours in B .

For example, in Figure 4.2, if we look at $K_{8,3}$ we see that $\lceil \frac{m}{4} \rceil + n = 2 + 3 = 5 \leq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil = 4 + 2 = 6$, but for $K_{8,6}$ we have $\lceil \frac{m}{4} \rceil + n = 2 + 6 = 8 \geq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil = 4 + 3 = 7$.

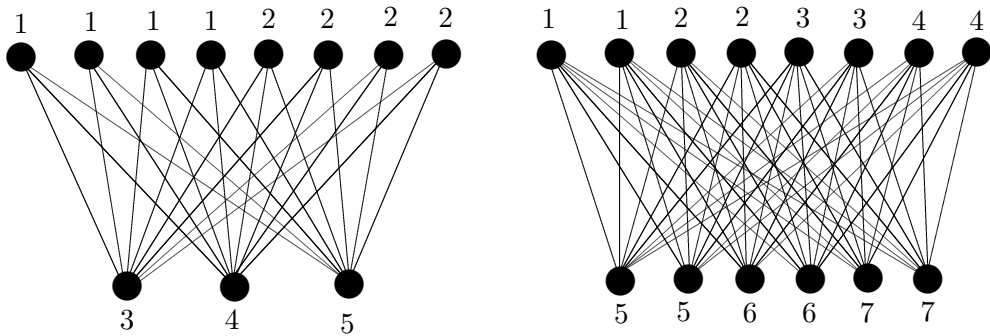


Fig. 4.2: $h_4(K_{8,3}) = 5$, $h_4(K_{8,6}) = 7$.

Therefore, $h_4(K_{m,n}) = \{ \min(\lceil \frac{m}{4} \rceil + n, \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil) \}$. Similarly, when $\lambda = 9$, depending on the graph, it may be best to use $\lceil \frac{m}{9} \rceil + n$ colours, $\lceil \frac{m}{4} \rceil + \lceil \frac{n}{2} \rceil$ colours, or $\lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil$ colours, as in $K_{9,1}$, $K_{4,2}$, and $K_{3,3}$, respectively.

In general, for each λ there will be several different colouring options for $K_{m,n}$, and the most appropriate colouring choice will depend on the size of m and n . Each colouring choice will allow colours to repeat t times in A , and s times in B . Since $m \geq n$, we will always have $t \geq s$. For example, for $\lambda = 9$, we could choose to use each colour twice in A ($t = 2$), and 4 times in B ($s = 4$), but choosing $t = 4$, $s = 2$ would always be better, since A is never smaller than B .

We also know that each colour pair can only be used up to λ times, so $t \times s \leq \lambda$. Therefore, our choice of s always depends on our choice of t , and so, we may choose $\{t = \lambda, s = 1\}$, $\{t = \lfloor \frac{\lambda}{2} \rfloor, s = 2\}$, \dots , or $\{t = \lfloor \frac{\lambda}{q} \rfloor, s = q\}$, for $1 \leq q \leq \lfloor \sqrt{\lambda} \rfloor$, $q \in \mathbb{Z}$. We do not need to consider $q > \lfloor \sqrt{\lambda} \rfloor$, because this would give us $s > t$. Therefore, $h_\lambda(K_{m,n}) = \left\{ \min \left(\left\lceil \frac{m}{\lfloor \frac{\lambda}{q} \rfloor} \right\rceil + \left\lceil \frac{n}{q} \right\rceil ; 1 \leq q \leq \lfloor \sqrt{\lambda} \rfloor, q \in \mathbb{Z} \right) \right\}$.

Note that for graphs where A is significantly larger than B , such as stars, we have $h_\lambda(K_{m,n}) = \lceil \frac{m}{\lambda} \rceil + n$, i.e. $q = 1$. However, when the sizes of A and B are closer together, the value of q becomes less clear.

Theorem 6: The λ -harmonious chromatic number of a path, P_n , is as follows:

Let $r \in \mathbb{Z}$ be determined by the inequality $\lambda \binom{2r-1}{2} < n-1 \leq \lambda \binom{2r+1}{2}$. Then

$$h_\lambda(P_n) = \begin{cases} 2r & \text{if } \lambda \text{ is even and } n-1 \leq \lambda \binom{2r}{2}, \\ & \text{or, if } \lambda \text{ is odd and } n-1 \leq \lambda \binom{2r}{2} - (r-1), \\ 2r+1 & \text{otherwise.} \end{cases}$$

Proof:

We gave the proof for the case where $\lambda = 1$ in Section 3. The proof for a general λ is quite similar, but we must take a few more factors into account.

To determine $h_1(P_n)$, we considered smallest k such that the complete graph on k vertices, K_k , had a subgraph with an eulerian path of length $n-1$. However, for a general λ , we are allowed to use each edge pair λ times. Therefore, instead of considering K_k , we will instead consider a graph on k vertices in which each pair of vertices is connected by λ edges. Let's call this new graph $H_{\lambda,k}$. For example, when $\lambda = 2$, $H_{2,5}$ would be K_5 with double edges (see Figure 4.3). In order to determine $h_\lambda(P_n)$, we will consider the smallest k such that $H_{\lambda,k}$ has a subgraph with an eulerian path of length $n-1$.

When λ is even, the degree of each vertex in $H_{\lambda,k}$ will be even, since the degree of each vertex in $H_{\lambda,k}$ is a multiple of λ . We know that K_k has $\binom{k}{2}$ edges, so $H_{\lambda,k}$ will have $\lambda \binom{k}{2}$ edges. Therefore,

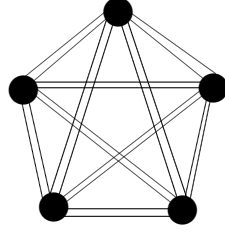


Fig. 4.3: $H_{2,5}$.

when λ is even, our k will be the smallest k such that $n - 1 \leq \lambda \binom{k}{2}$, i.e. the length of P_n must be less than or equal to the number of edges in $H_{\lambda,k}$.

When λ is odd and k is odd, the degree of each vertex in $H_{\lambda,k}$ will also be even, since the degree of each vertex in $H_{\lambda,k}$ is a multiple of $k - 1$. Therefore, in this case, k will also be the smallest k such that $n - 1 \leq \lambda \binom{k}{2}$.

When λ is odd and k is even, the degree of each vertex in $H_{\lambda,k}$ will be odd. Therefore, in order to create an eulerian path, we must delete at least $\frac{k}{2} - 1$ edges, in order to be left with at most two vertices of odd degree. Therefore, our k will be the smallest k such that $n - 1 \leq \lambda \binom{k}{2} - (\frac{k}{2} - 1)$.

The theorem follows, with $r = \frac{k}{2}$.

Theorem 7: The λ -harmonious chromatic number of a cycle, C_n , is as follows: Let k be the least integer such that $n \leq \lambda \binom{k}{2}$. Then

$$h_\lambda(C_n) = \begin{cases} k & \text{if one of the following four conditions hold:} \\ & i) \lambda \text{ is even and } n \neq \lambda \binom{k}{2} - 1, \\ & ii) \lambda \neq 1, \lambda \text{ is odd, } k \text{ is odd, and } n \neq \lambda \binom{k}{2} - 1, \\ & iii) \lambda = 1, k \text{ is odd, and } n \neq \lambda \binom{k}{2} - i \text{ for } i = 1, 2, \\ & iv) \lambda \text{ is odd, } k \text{ is even, and } n \neq \lambda \binom{k}{2} - i \text{ for } i = 0 \dots \frac{k}{2} - 1, \\ k + 1 & \text{otherwise.} \end{cases},$$

Proof:

We will determine the λ -harmonious chromatic number for C_n by extending the proof for $\lambda = 1$ given by Lee and Mitchem [4], which we outlined in Section 3. We will do this by considering $H_{\lambda,k}$, which is defined the same as in Theorem 6.

Let k be the smallest integer such that $H_{\lambda,k}$ has a subgraph with an eulerian cycle of length n . When λ is even, or when λ is odd and k is odd, then each vertex in $H_{\lambda,k}$ will have even degree.

We know that a connected graph G has an eulerian cycle if and only if each vertex in G has even degree. Therefore, in these two cases, we can create a subgraph of $H_{\lambda,k}$ with an eulerian cycle of length n by either deleting zero edges, or by deleting the edges from a cycle in $H_{\lambda,k}$. If $n = \lambda \binom{k}{2} - 1$ we won't be able to do this, since no cycles of length 1 exist. If $n = \lambda \binom{k}{2} - 2$, we can delete a cycle of length 2 in our graph if $\lambda \neq 1$ (i.e. $H_{\lambda,k}$ always has multiple edges between vertices when $\lambda > 1$). Therefore, in these two cases, if $\lambda \neq 1$, then $h_\lambda(C_n) = k$, where k is the least integer such that $n \leq \lambda \binom{k}{2}$, i.e. the length of C_n must be less than or equal to the number of edges in $H_{\lambda,k}$. For the case where $\lambda = 1$ and k is odd, see Theorem 3.

If $n = \lambda \binom{k}{2} - 1$, we cannot create this eulerian subgraph, but we can create it for $H_{\lambda,k+1}$. As in Theorem 3, we can create an eulerian subgraph of $H_{\lambda,k+1}$ of length n by deleting the edges from a P_4 in $H_{\lambda,k}$ that uses four distinct vertices, and then joining the end vertices of P_4 to a new edge v not in $H_{\lambda,k}$. Therefore, in these two cases, if $n = \lambda \binom{k}{2} - 1$, then we will have $h_\lambda(C_n) = k + 1$.

When λ is odd and k is even, then all vertices in $H_{\lambda,k}$ will have odd degree. Therefore, if $n \neq \lambda \binom{k}{2} - i$ for $i = 0 \dots \frac{k}{2} - 1$, then we will have $h_\lambda(C_n) = k$, since we can create a subgraph of $H_{\lambda,k}$ with an eulerian cycle of length n in the same way as the k even case of Theorem 3. If $n = \lambda \binom{k}{2} - i$ for $i = 0 \dots \frac{k}{2} - 1$, then we will have $h_\lambda(C_n) = k + 1$, since we can't create a subgraph of $H_{\lambda,k}$ with an eulerian cycle of length n , but, we can create it for $H_{\lambda,k+1}$ as in the k even case of Theorem 3.

Theorem 8: Let $q = h_\lambda(K_{n-1,1})$. If $q > 4$, then $h_\lambda(W_n) = q$.

Proof:

A wheel on n vertices consists of a single vertex, called the hub, and a cycle of length $n - 1$, such that the hub is adjacent to all vertices of C_{n-1} . Therefore, each wheel contains a star $K_{n-1,1}$ and a cycle C_{n-1} . Let $h_\lambda(K_{n-1,1}) = q$. Since W_n contains $K_{n-1,1}$ as a subgraph, we know that $h_\lambda(W_n) \geq q$. When $q > 4$ we will show that, in fact, $h_\lambda(W_n) = q$. We will do this by considering $H_{\lambda,q-1}$, which is defined the same as in Theorem 6. We will show that $H_{\lambda,q-1}$ has a subgraph with an eulerian cycle of length $n - 1$ such that each vertex in the cycle is only used at most λ times. Therefore, the C_{n-1} of the wheel can be harmoniously coloured with $q - 1$ colours, and this leaves one colour left over for the hub, which will be adjacent to each colour at most λ times. Hence, $h_\lambda(W_n) = q$.

From Theorem 5, we know that $h_\lambda(K_{n-1,1}) = \lceil \frac{n-1}{\lambda} \rceil + 1$. Since we require q colours to colour our star, this means that the $n - 1$ end vertices of $K_{n-1,1}$ (i.e. the vertices of degree one) must partition into $q - 2$ groups of size λ and one group of size less than or equal to λ , $\lambda - i$ (see Figure 4.4).

Now, we can create a subgraph of $H_{\lambda,q-1}$ which corresponds to the partitions of the $n - 1$ end

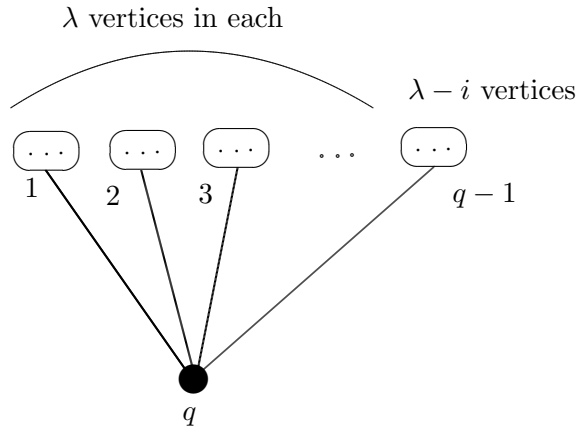


Fig. 4.4: $h_\lambda(K_{n-1,1}) = q$.

vertices of our star, i.e. $q - 2$ vertices of $H_{\lambda,q-1}$ will have degree 2λ and one vertex will have degree $2(\lambda - i)$. Such a subgraph will have an eulerian cycle, since each vertex will have even degree. We can create this subgraph in the following way. Take $q - 1$ vertices and give them all distinct colours $\{1, 2, \dots, q - 1\}$. Now, put λ edges between the vertices coloured 1 and 2, λ edges between the vertices coloured 2 and 3, and so on until you put λ edges between the vertices coloured $q - 3$ and $q - 2$. Now, put $\lambda - i$ edges between the vertices coloured $q - 2$ and $q - 1$, $\lambda - i$ edges between the vertices coloured $q - 1$ and 1, and i edges between the vertices coloured 1 and $q - 2$. We are left with a subgraph of $H_{\lambda,q-1}$ with $n - 1$ edges such that $q - 2$ vertices have degree 2λ , and one has degree $2(\lambda - i)$ (see Figure 4.5).

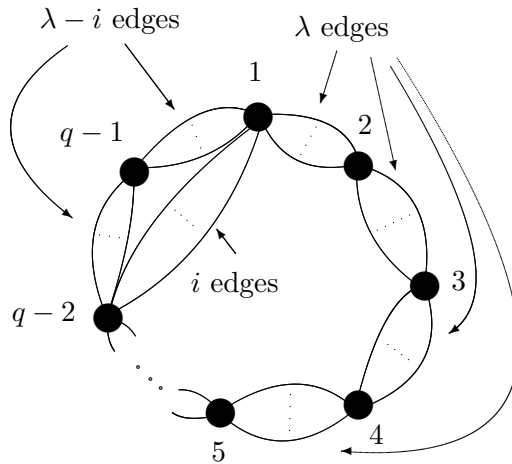


Fig. 4.5: An eulerian subgraph of $H_{\lambda,q-1}$.

Therefore, we can colour our wheel W_n by using the colouring of this eulerian cycle to colour

C_{n-1} and using a distinct colour for the hub. Our colouring ensured that each colour on our cycle was only used at most λ times, so our q -colouring is λ -harmonious, and it is the best we can possibly do, since $h_\lambda(K_{n-1,1}) = q$. Therefore, $h_\lambda(W_n) = q$.

Note that when $q = 4$, $q = 3$, or $q = 2$ we may not have $h_\lambda(W_n) = q$. For example, $h_4(K_{4,1}) = 2$, but we cannot properly colour W_5 with 2 colours (see Figure 4.6). Also, $h_3(K_{8,1}) = 4$, but Theorem 7 tells us that $h_3(C_8) = 4$, so we need at least 5 colours to colour W_9 . Similarly, $h_2(K_{5,1}) = 4$, but $h_2(C_5) = 4$, so we need at least 5 colours to colour W_6 .

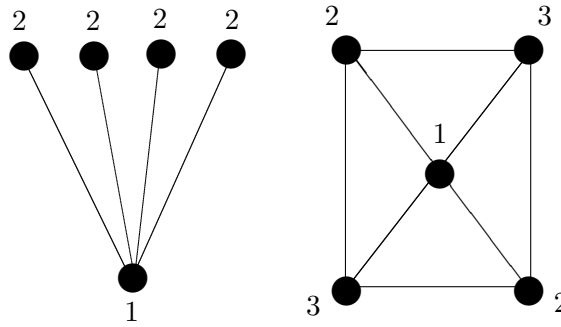


Fig. 4.6: $h_4(K_{4,1}) = 2$, but $h_4(W_5) = 3$.

5. CONCLUSION

Although the harmonious chromatic number has been found for several families of graphs, determining $h(G)$ is, in general, NP-complete. In the appendix of [3], David S. Johnson shows that the independent set problem can be transformed in polynomial time into the harmonious colouring problem. Since we know that the independent set problem is NP-complete, this tells us that the harmonious colouring problem is also NP-complete.

We can formally state the λ -harmonious colouring problem as follows: Given a graph G and a positive integer $k \leq |V(G)|$, can G be λ -harmoniously coloured with k colours? In other words, does there exist a proper vertex colouring of G with k colours such that each edge colour pair occurs at most λ times? We suspect that the λ -harmonious colouring problem is also NP-complete, but we haven't been able to prove it yet.

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