The Evolution of Grötzsch's Theorem

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ABSTRACT

In 1959, the German mathematician H. Grötzsch proved that any planar graph without triangles can be coloured without same colour vertices being adjacent using only three colours. Grötzsch used vertices that had the potential to be coloured by any of his three colours. New colouring techniques, on the other hand, assign lists of potential colours to each vertex (it could be possible to have the colour green in your graph, but have a vertex without green in its list). This technique may sound more complicated, but it has reduced the complexity of Grötzsch's proof significantly. We will examine how using different proof techniques can reduce the difficulty of a proof, and show how Grötzsch's Theorem has led to many generalized three colour theorem results.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. James Preen, for all of his encouragement, help, and support while making this thesis. I would also like to thank him for being such a great professor. His enthusiasm made graph theory, as well as every other course, a really enjoyable experience. I would also like to thank him for always having time to answer my many many questions!

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CONTENTS

LIST OF FIGURES

1. INTRODUCTION

Graph colouring originated in the mid-nineteenth century, when mathematicians began to ask the question, "Can a geographical map always be coloured using 4 colours or less?" In 1878, A. Cayley represented the four colour problem using a system of vertices and edges, where the vertices represented countries, and there was an edge between vertices if they shared a border on the map. Cayley therefore restructured the four colour problem by asking, "Can any planar graph be coloured using 4 colours or less?" (figure 1.1) This seemingly simple problem was a lot harder to prove than many mathematicians had initially expected. Sir A. Kempe published a faulty proof in 1879, in which the mistake was discovered 11 years later. It was not until almost a century later, in 1976, that K Appel, W. Haken, and J. Koch finally proved the 4 colour theorem, using 1476 unavoidable configurations. Since then, the proof has been improved, but it still requires extensive computer calculations to confirm [1].

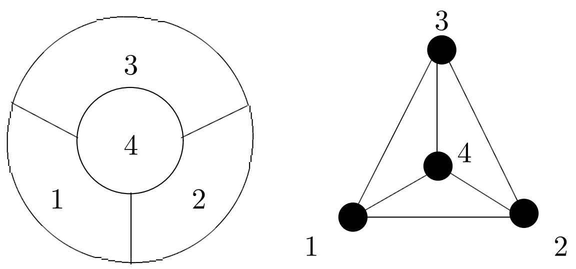

Fig. 1.1: Four Colour Problem.

In 1959, the German mathematician H. Grötzsch created a three colour theorem, which states that any planar graph with girth at least 4 (triangle-free) can be properly coloured using at most 3 colours. We need the graph to be both planar and triangle-free, because there exist planar graphs that are not 3-colourable (figure 1.1) and triangle-free graphs that are not 3-colourable (figure 1.2). Grötzsch's original proof was quite complex, so mathematicians have since simplified his proof. One of those people was L. Kowalik [8] who simplified Grötzsch's result and used it to create an algorithm. Following this algorithm, one can 3-colour the vertices of any planar graph which has girth at least 4.

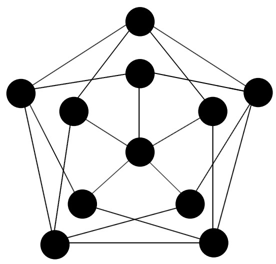

Fig. 1.2: Grötzsch's graph: non-planar and triangle-free.

In 2003, C. Thomassen [10] also improved Grötzsch's result when he published a short and elegant proof of Grötzsch's theorem using a technique called list colouring. Thomassen proved that every planar graph of girth at least 5 is 3-list-colourable and used this result to conclude that every triangle-free planar graph is 3-colourable. Note that Thomassen did not solely use list colouring to prove Grötzsch's result, because there exist planar graphs with girth 4 that are not 3-list-colourable [5]. Therefore, although it is true that all triangle-free planar graphs are 3-colourable, we know that there exist triangle-free planar graphs that are not 3-list-colourable.

We will examine Kowalik and Thomassen's proofs in Chapters 3 and 4, respectively, by taking an in-depth look at a few particular cases. It may be informative to read Kowalik and Thomassen's proofs alongside these ones. We hope that examining these proofs in more detail will help novice graph colourers better understand the techniques used in graph colouring literature.

2. TERMINOLOGY

For additional terminology, see [2]. We will denote

path - an alternating sequence of distinct vertices and edges that begins and ends with a vertex. For example, $P : v_1 v_2 v_3$ would be a path with a length of 3.

cycle - a closed path where the first and last vertex are the same. For example $H = v_1 v_2 v_3 v_1$ would be a cycle with a size of 3 (also known as a 3-cycle or a triangle).

planar - a graph that can be drawn without edges crossing.

$|\boldsymbol{V(G)}|$ **-** the number of vertices in a graph G.

face - a region in a planar graph that is surrounded by a cycle, such that there are no edges reaching from the cycle into the region. Note that the exterior of a graph is also a face, known as the outer face (see boundary).

boundary - the vertices and edges that border a face. Let C denote the boundary of the outer face. $|C|$ refers to the size of the face C.

chord - a chord of a cycle H is an edge joining 2 vertices of H that are not adjacent in H.

opposite vertices - if a cycle H has size 6, then two vertices on H are opposite if there are 3 edges of H between them. For example, in figure 2.1, all opposite vertices share the same colour. Note that we will use numbers to denote colours.

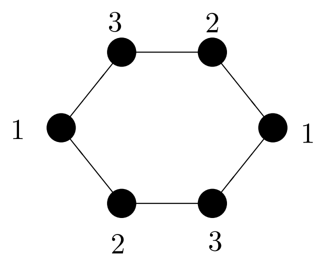

Fig. 2.1: Opposite vertices.

degree - the degree of a vertex v, or $\deg(v)$, refers to the number edges incident to v. Since we are dealing with simple graphs, the degree of v refers to the number of neighbours v has.

distinct - a cycle H is distinct from another cycle J if H contains a vertex or an edge not in J.

girth - the size of a graph's shortest cycle, (If a graph has girth 5 then it does not contain any 3-cycles or 4-cycles.) A triangle-free graph has girth at least 4.

identification of vertices - merge two vertices, x and y, into one new vertex i, where i shares all of x and y's neighbours, but does not have any double edges.

disconnected graph - a graph is disconnected if there exist two vertices such that there is no path that has those two vertices as endpoints.

separating - a cycle S is separating in a connected graph G if deleting all vertices in S disconnects the graph G.

$\boldsymbol{G - x}$ **-** refers to G with a vertex x removed along with all of x's edges.

$\boldsymbol{G[H]}$ **-** a subgraph induced by H in G, that is, G with all vertices and edges not in H removed.

int($\boldsymbol{H}$**) -** the interior vertices of H, where H is a cycle in a planar graph.

proper colouring - no two adjacent vertices share the same colour.

r**-colourable -** G can be properly coloured using only r colours.

safe - a 3-colouring of the outer boundary C is called safe if C has size at most 6 and has the following properties: if C has size less than 6, then any colouring of C will suffice. If C has size 6, then the sequence of successive colours on the cycle is neither $(1, 2, 3, 1, 2, 3)$ nor $(3, 2, 1, 3, 2, 1)$. In other words, there must be at least one set of opposite vertices on C that do not share the same colour. Note that figure 2.2 is safely coloured, but figure 2.1, is not.

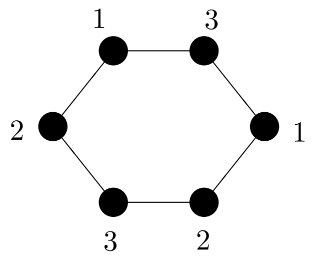

Fig. 2.2: Safe colouring.

list colouring - a type of graph colouring in which each vertex v is assigned a list $L(v)$ of potential colours (see figure 2.3). One must assign a colour to each vertex such that each vertex is given a colour from its list. In order for a graph to be properly list coloured, no two adjacent vertices can receive the same colour. Note that r colouring is a special case of list colouring in which each vertex is assigned the same list of r colours.

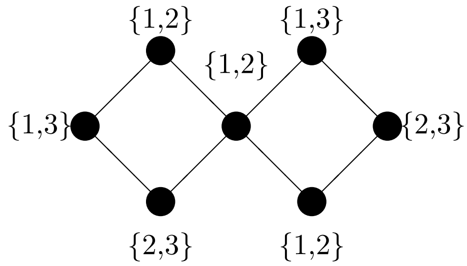

Fig. 2.3: A 2-colourable, but not 2-list-colourable graph.

r-list-colourable - each list has at most r colours and it has a proper list colouring regardless of how one assigns colours to each vertex's list. For example, figure 2.3 is 2-colourable, but it is not 2-list-colourable. Given the lists shown, if the middle vertex receives the colour 1, then the bottom right must receive the colour 2, the top right must receive the colour 3, and so then the far right vertex must be coloured 2 or 3, but no matter what colour we choose there will be two adjacent vertices with the same colour, which is not a proper list colouring. Similarly, if we instead colour the middle vertex 2, we will encounter a similar problem on the left, by symmetry. Therefore, the graph in figure 2.3 is not 2-list colourable, because in order for it to be 2-list-colourable it must have a proper colouring for every possible list assignment, but since there exist list assignments (figure 2.3) that do not work, the graph is not 2-list-colourable.

list-r vertex - vertices that have r colours in their list. For example, if a vertex v_1 had a list: $\{1, 3\}$ then v_1 would be a list-2 vertex.

3. KOWALIK'S PROOF OF GRÖTZSCH'S THEOREM

Grötzsch's theorem states that any planar graph with girth at least 4 (triangle-free) can be properly coloured using at most 3 colours. Kowalik [8] simplified Grötzsch's result and formatted his proof in such a way that it can be treated as a scheme for an algorithm that 3-colours the vertices of planar graphs which have girth at least 4. Note that we are assuming that the graphs we deal with have a fixed structure, i.e. if the outer boundary has size 5, we do need to consider whether or not the graph could be redrawn to have an outer boundary size of 4.

Theorem 1: Any planar graph with girth at least 4 (triangle-free) is properly 3-colourable. Moreover, if the outer boundary of G is a cycle of size at most six, then any safe 3-colouring of the boundary cycle C can be extended to a 3-colouring of G.

Proof: Let G be a triangle-free planar graph. We will proceed by induction on n, the number of vertices in G. Our induction hypothesis will be: G can be properly 3-coloured if it has $n-1$ vertices or less. If G is disconnected, we will apply induction separately to each component of G. By the definition of a safe cycle, one of these two conditions will hold:

1. All vertices in G are uncoloured.
2. The boundary C is safely coloured, the induced graph $G[C]$ is properly coloured, and all vertices not in C are uncoloured (recall in order to have a safe colouring $|C| \leq 6$).

Note that we are requiring that if the outer boundary is coloured, then it is safely coloured, because this stronger condition makes it easier for us to use induction in certain cases, like in the upcoming case 3

We will first proceed by considering three basic cases. We will then use these results to deal with graphs that have faces of size 6 or greater (Case 4), faces of size 4, and finally faces of size 5. We do not have to consider faces of size 3 since our graph does not contain any triangles. We will use this property to help us eliminate each case. After each case is exhausted we will assume it can no longer apply to G.

Case 1: Suppose G has an uncoloured vertex v of degree at most 2.

We can delete v and apply induction to the rest of the graph. When we add v back into our coloured graph G, it will be adjacent to at most two colours, since it only has at most two neighbours. This means that of the three colours being used in G, there will be at least one colour that is not being used on v's neighbours; we will use this colour to colour v. (If $v \in C$ we know that C is uncoloured since v is uncoloured, and we proceed in the same way.) For example, in figure 3.1, v is uncoloured and has degree 2, so we can delete it, apply induction to P_1, apply induction to P_2, and then add v back into G.

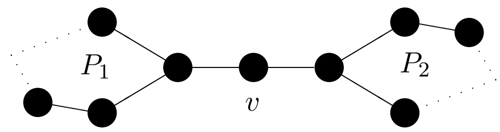

Fig. 3.1: G has an uncoloured vertex with degree at most 2.

Case 2: Suppose C has a chord.

Since C has a chord, we know C has at least 6 vertices, since G is triangle-free. Also, if C is coloured, we know that this chord is not adjacent to two vertices with the same colour, because $G[C]$ is properly coloured by definition. This chord splits C into two cycles, P_1 and P_2. For example, in figure 3.2, $P_1 = x_1x_2x_3x_6x_1$ and $P_2 = x_3x_4x_5x_6x_3$. Therefore, we can first remove everything in G except for $P_2 \cup \mathrm{int}(P_2)$, and colour $P_2 \cup \mathrm{int}(P_2)$ by induction. We can then remove everything in G except for $P_1 \cup \mathrm{int}(P_1)$, and colour $P_1 \cup \mathrm{int}(P_1)$ by induction, using the colours assigned by the first colouring (figure 3.2). We can proceed in this manner regardless of whether C is coloured or not. Note that we are using circled vertices to represent coloured vertices.

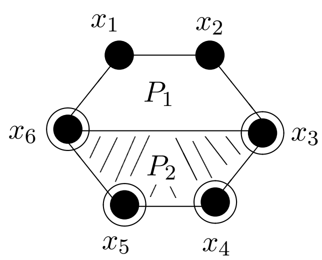

Fig. 3.2: C has a chord.

Case 3: Suppose G contains an uncoloured vertex x joined to more than one coloured vertex.

Note that x has degree at least 3 since Case 1 takes care of uncoloured vertices of degree one or two. We also know that coloured vertices exist in G, and these vertices must be in C, so C has at most 6 vertices. Therefore, x's coloured neighbours are in C, but $x \notin C$ since it is not coloured. x can only have 2 or 3 coloured neighbours, since G is triangle-free. If x has three coloured neighbours, then C has size 6, since there are no triangles in G (figure 3.3). However, x cannot have three differently coloured neighbours, since C is safe (figure 2.2). Therefore, we can colour x using this spare colour. We will now proceed in a similar manner as in Case 2. $G[V(C) \cup \{x\}]$ splits C into several cycles. The number of cycles depends on how many neighbours x has. For example, in figure 3.3 there are 3 cycles: $P_1 = x_1x_2xx_6x_1$, $P_2 = x_3x_4xx_2x_3$, $P_3 = x_5x_6xx_4x_5$. We know that these cycles are safely coloured, since C was safely coloured and since we properly coloured x. Therefore, we can proceed by induction by considering each cycle separately, as we did in Case 2. For example, in figure 3.3, we will first remove everything in G except for $P_1 \cup \mathrm{int}(P_1)$, and colour $P_1 \cup \mathrm{int}(P_1)$ by induction. We will then remove everything in G except for $P_2 \cup \mathrm{int}(P_2)$, and colour $P_2 \cup \mathrm{int}(P_2)$ by induction. Finally, we will remove everything in G except for $P_3 \cup \mathrm{int}(P_3)$, and colour $P_3 \cup \mathrm{int}(P_3)$ by induction.

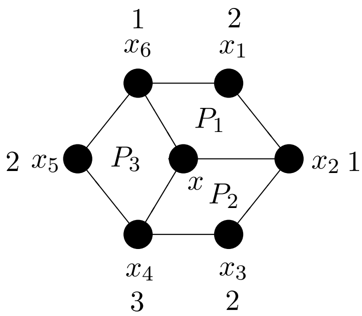

Fig. 3.3: An uncoloured vertex x joined to more than one coloured vertex.

In summary, the properties of G after Case 2 are:

- Uncoloured vertices have degree at least 3.
- There is no chord in C.
- Uncoloured vertices have at most 1 coloured neighbour.

We will now introduce a claim that will help us take care of the upcoming cases.

Claim 1: When G contains a separating cycle S distinct from C, where $4 \leq |V(S)| \leq 6$, then we can complete the proof of theorem 1 by induction.

Let S be our separating cycle distinct from C. Since S is distinct from C, S contains at least one uncoloured vertex y. Now let $A = G - \text{int}(S)$ and $B = S \cup \text{int}(S)$. If $|V(S)| = 6$, S may or may not contain a chord. If S does contain a chord, let that chord be part of A; if S does not contain a chord, put a chord in A between y and its opposite vertex. Now that we have a chord, we can complete the proof using the technique we used in Case 2. We will colour G by first applying our induction hypothesis to A. We know S will be safely coloured, since it contains a chord. Therefore, using S as the boundary cycle, we can now apply the induction hypothesis again to colour B, which completes our colouring of G (figure 3.4).

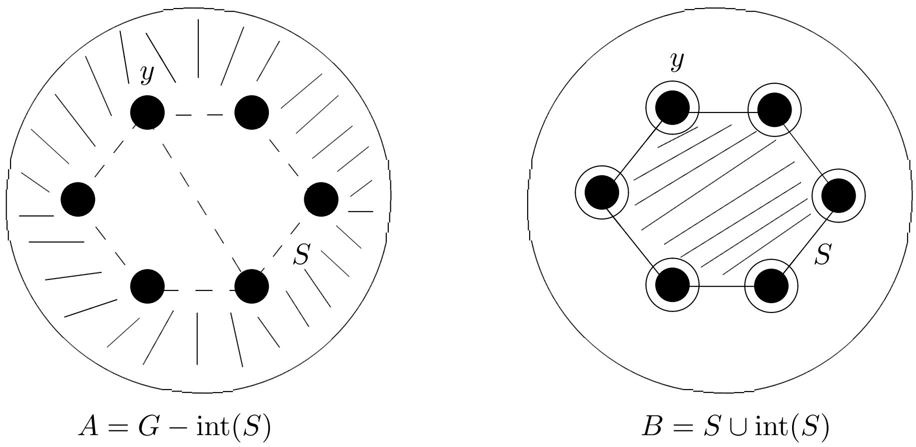

Fig. 3.4: G contains a separating cycle, S.

Case 4:

This case will deal with all graphs that contain a face T with size at least 6, where T has at least one uncoloured vertex, $(T : t_1t_2 \dots t_kt_1,\ k \geq 6)$. We know that if C is coloured, then it might have size 6, so this case will take care of all graphs with a face of size at least 7 and all graphs that contain an uncoloured face of size 6. Therefore, once this case is exhausted, the only face that could have size larger than 5 is the coloured boundary C, where $|V(C)| = 6$. Let the uncoloured vertex be t_1. Now t_2 or t_k must be uncoloured as well, because otherwise we could just use Case 3, so let t_2 be uncoloured. We want to identify t_1 with t_3 in order to make a smaller graph, which will allow us to use the induction hypothesis, but first we must take care of the cases where identifying t_1 with t_3 creates a chord or a triangle, Subcases 4.1 and 4.2, respectively.

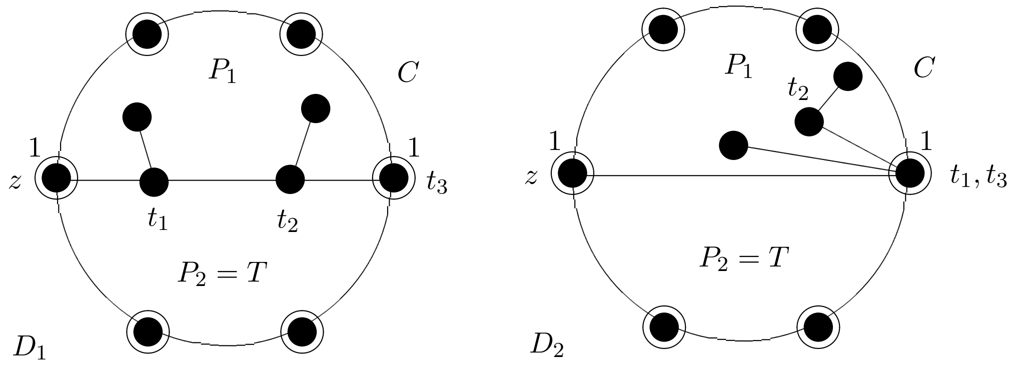

Fig. 3.5: Identifying t_1 with t_3 creates a chord in C: $P_2 = T$.

Subcase 4.1: Assume t_3 is coloured, and t_1 has a coloured neighbour z.

Since t_3 and z are both coloured, we know that they must be in C. We also know that the degree of t_1 and t_2 are both greater than two, because otherwise we could just use Case 1. However, t_1 and t_2 do not share any more neighbours with C, or else we could just use Case 3. Now $G[C \cup \{t_1, t_2\}]$ splits G into two cycles, let's call them P_1 and P_2. Without loss of generality, let us assume $|P_1| \leq |P_2|$, where $|P_1|$ is the number of vertices in P_1. We know $|C| \leq 6$, since it is coloured. Therefore P_1 has at most 6 vertices. We know this because we know P_1 and P_2 both contain t_1, t_2, t_3, and z (recall t_3 and z are both in C). Therefore, $|C| = 6 = (|P_1| - 2) + (|P_2| - 2) - 2$ (we subtract t_1 and t_2 from both, and also subtract t_3 and z since they are in both P_1 and P_2). Now we can see P_1 must have size 6 or less, because if it had size 7, P_2 would have size 5 ($|C| = 6 = (7 - 2) + (|P_2| - 2) - 2$), which would make P_1 larger than P_2, which is a contradiction (D_1 in figure 3.5).

We also know that t_1 and t_2 each have a third edge, but do not share any other neighbours with C, so their third edge must be connected to a vertex inside either P_1 or P_2, or both. If P_1's boundary is a separating cycle then we can complete the proof by using Claim 1, since $|P_1| \leq 6$. Note that we want to use Claim 1 here, because identifying t_1 with t_3 would create a chord in C, which could result in having two vertices with that same colour being adjacent, which would be an improper colouring (D_2 in figure 3.5).

For example, in D_1 in figure 3.5, z and t_3 are both coloured 1, so identifying t_1 with t_3 would create an edge between two vertices of same colour (D_2 in figure 3.5). Note that $P_2 = T$ in this example.

If P_1 is not a separating cycle, then P_2 must be a separating cycle, but we must ensure that it has size at most 6 in order to use Claim 1. If P_1 is not separating, then the face T cannot be contained inside P_2, because t_2's third edge would not have anywhere to go (D_1 in figure 3.6). It cannot go inside P_1 or else P_1 would be a separating cycle, and t_2 cannot create a chord with a vertex of T or be adjacent to a vertex inside of T, since T is a face. Therefore, P_1 must equal T. (D_2 in figure 3.6). Since $|P_1| \leq 6$ by definition, this means that P_1 has size 6 since $P_1 = T$, and P_2 must also have size 6, since $|P_1|$ is less than or equal to $|P_2|$ and the coloured boundary C has size at most 6. Therefore, P_2 is separating, and has size 6, so we can colour G by using Claim 1.

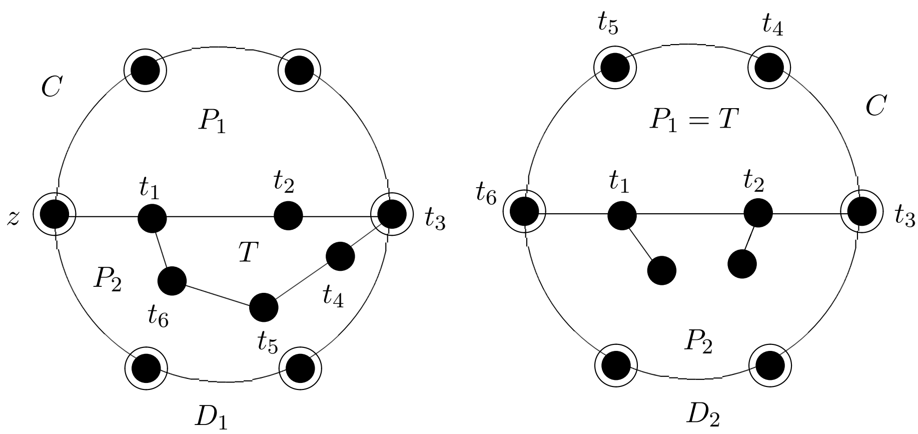

Fig. 3.6: Identifying t_1 with t_3 creates a chord in C: $P_1 = T$.

Subcase 4.2: Assume there is a path $t_1y_1wt_3$.

This will take care of all the cases where identifying t_1 with t_3 creates a triangle. We know $t_2 \notin \{w, y_1\}$ since G has girth at least 4 (figure 3.7). Therefore, there is a cycle $S = t_1y_1wt_3t_2t_1$. The degree of t_2 must be more than two, because otherwise we could just use Case 1 (recall t_2 is uncoloured). This means that t_2's third neighbour must enter into S, since T is a face. Therefore, S is separating, and we can apply Claim 1. For example, S could take any of the 3 forms shown in figure 3.7, so identifying t_1 with t_3 in any of these graphs would create a triangle (figure 3.8).

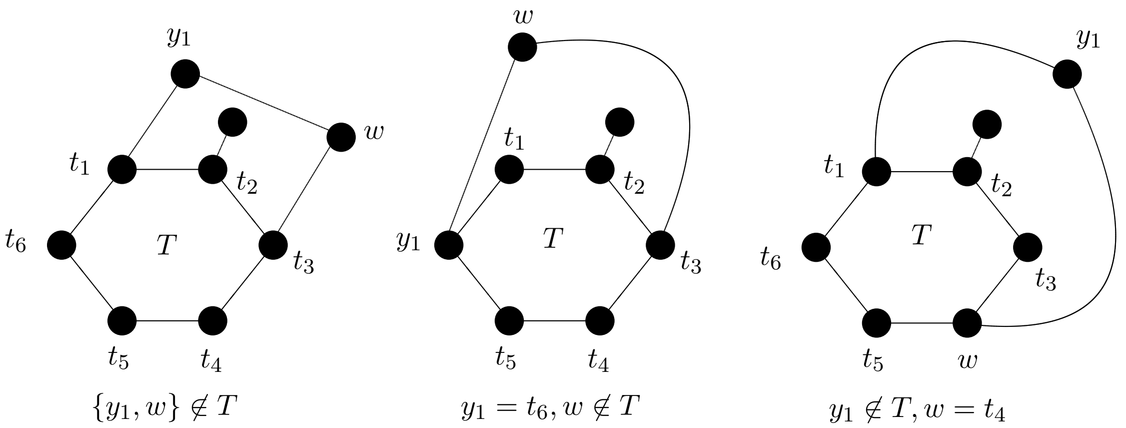

Fig. 3.7: Situations where identifying t_1 with t_3 would create a triangle in G.

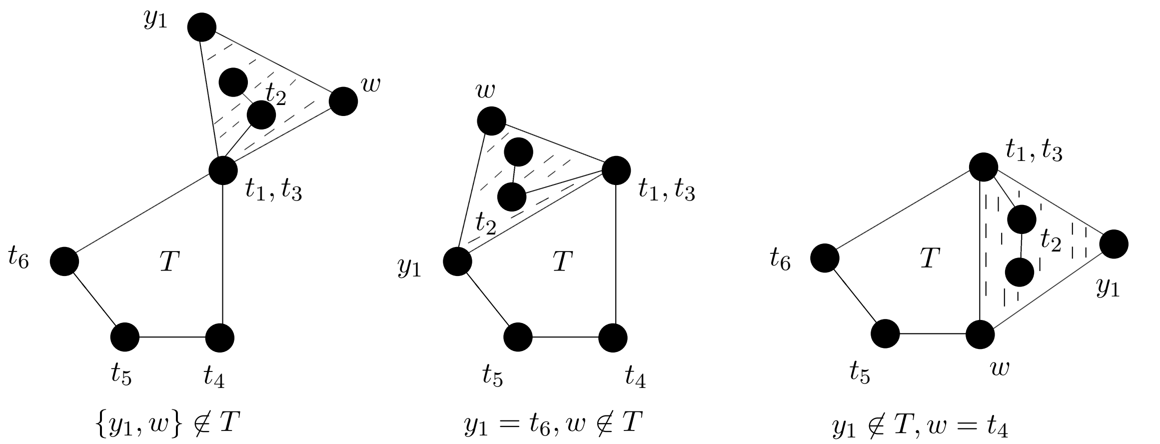

Fig. 3.8: Triangles that could potentially occur.

Subcase 4.3: Identify t_1 with t_3 to create a smaller graph, M.

We know this identification will not create a chord through C or a triangle, since we took care of those possibilities in Subcases 4.1 and 4.2, respectively. When we identify t_1 with t_3, the merged vertex t_1, t_3 will take on t_3's colour. M will not have 2 adjacent vertices with the same colour since t_1 and t_2 are uncoloured in G. M has $n-1$ vertices, so we can colour it using the induction hypothesis (figure 3.9). After we apply induction, we will separate t_1 and t_3 once more to recreate G. t_1 and t_3 will not have any additional neighbours that t_1, t_3 did not have. Therefore, the colouring will still be proper, as t_1 and t_3 will take t_1, t_3's colour, and their neighbours will not have this colour, since t_1, t_3's neighbours did not have this colour. This takes care of all of the graphs that contain a face of size at least 7 or an uncoloured face of size 6.

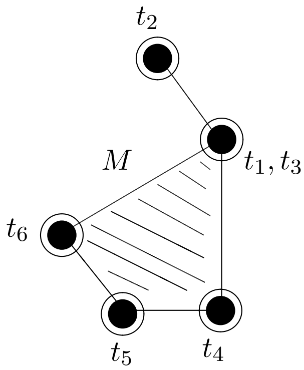

Fig. 3.9: Identifying t_1 with t_3.

Kowalik uses similar techniques to prove the theorem for graphs that contain a face with size 4, distinct from C. After this case is exhausted, he assumes that all faces in the interior of C must have size 5, because all other face sizes were considered. In order to take care of this case, Kowalik introduces an additional theorem that uses *discharging* to prove that G must contain a 5-cycle, and that all vertices in this cycle have degree 3, except for one vertex, which has degree at most 5.

Discharging is a proof technique used to show that a certain structure in a graph must exist. It uses Euler's formula, which relates the number of vertices V, edges E, and faces F, in a connected planar graph such that $V-E+F=2$. It also uses the fact that in planar graphs there is a relation between the number of edges, the total degree of the graph, and the sum of all the face sizes: $2E = \sum_{v\in V} \deg(v) = \sum_{f\in F} |f|$. A charge is generally given to the vertices and faces of the graph, using Euler's Formula, and then a set of discharging rules redistributes the charge, without changing the total charge of the graph. For example, Kowalik assumed that a minimal counterexample existed. He then put a charge of $\deg(v)-4$ on every vertex v and a charge of $|f|-4$ on every face f. He also gave the outer face an additional charge of 7. Kowalik then used Euler's Formula to find out that the total charge of his graph was -1:

$$\begin{aligned}
\sum_{v\in V}(\deg(v)-4)+\sum_{f\in F}(|f|-4)+7 &= \sum_{v\in V}\deg(v)-4V+\sum_{f\in F}|f|-4F+7\\
&= 2E-4V+2E-4F+7\\
&= 4E-4V-4F+7\\
&= -4(-E+V+F)+7\\
&= -4(2)+7\\
&= -1
\end{aligned}$$

Kowalik then redistributed the charge from vertices to faces in such a way that v sent a charge of $\frac{\deg(v)-4}{\deg(v)}$ to each face incident with v. This redistribution should not have changed the total charge of the graph, but it turned out that doing so resulted in the total charge becoming nonnegative, which was a contradiction.

Kowalik completes the proof by identifying several vertices, but first he considers 8 additional subcases, which ensure that this identification does not create a triangle or a chord through C. This completes the proof.

4. LIST-COLOURING PROOF OF GRÖTZSCH'S THEOREM

Thomassen [10] uses a series of configurations, and then list colouring to prove that any planar graph with girth at least 5 is 3-list-colourable. We will first give an outline of this proof, and then we will prove that this result also implies that any triangle-free planar graph is 3-colourable.

Theorem 2: Let G be a planar graph of girth at least 5. Let P be a 3-coloured path or cycle $P : v_1v_2 \dots v_q, 1 \leq q \leq 6$, such that all vertices of P are on the outer boundary C. For each vertex v in G, let $L(v)$ be a list of colours. If v is not on the outer boundary C, then $L(v)$ has three colours, so v is a list-3 vertex. If v is on the outer boundary C, but is not in P, then $L(v)$ has at least 2 colours. Assume furthermore that there is no edge joining vertices whose lists have at most two colours, except for the edges in P (no list-2 vertices are adjacent to other list-2's or to any coloured vertices (list-1's), and coloured vertices cannot be adjacent to each other, unless they are both in P). Then the colouring of P can be extended to a proper 3-list-colouring of G.

Proof: We will proceed by induction on the number of vertices in G. We will rule out specific cases, and then use these cases as tools to help us complete the proof. When we manipulate the graph, we must always check to make sure the resulting graph's structure still satisfies the induction requirements (i.e. P, the precoloured part of the outer boundary, has at most six vertices, and no list-2 vertices are adjacent to other list-2's or to any coloured vertices. Recall that list-2 vertices may only exist in $C - P$. Also, coloured vertices cannot be adjacent to each other, unless they are both in P.) Using techniques similar to those used in Chapter 3 and the upcoming Case 6 of Chapter 4, Thomassen proves:

(a) G is connected.

(b) G has no cut-vertex. In other words, there does not exist a vertex in G such that $G - v$ is a disconnected graph.

(c) No chord in the outer boundary C is made by an edge of P. Therefore, we may choose a notation for the outer boundary $C : v_1v_2 \dots v_qv_{q+1} \dots v_kv_1$, such that $\{v_1, v_2, \dots, v_q\} \in P$.

(d) P is a path, and $q + 3 \leq k$. (In other words, the outer boundary C contains at least three vertices not in P.)

(e) C has no chord.

(f) If S is a cycle in G with at most six vertices and is distinct from C, then the interior of S is empty.

Please note that in the diagrams, the path P is usually of size 6, but in general, P may be smaller. We will denote list-2 vertices as squares, and list-3 vertices as triangles. Circled vertices will represent vertices that are originally coloured in G, and vertices with a square around them will represent vertices that receive a colour by induction, or by an additional colouring.

Case 5: G has no path of the form $v_j u v_i$ where u is in $\mathrm{int}(C)$ and v_j, v_i are in C, except possibly when P has size 6 and the path is of the form $v_4 u v_7$ or $v_3 u v_k$.

Let us use the path $v_j u v_i$ to divide G into two graphs P_1 and P_2 such that P_1 has more vertices of P in it than P_2 does, and as a result, $|V(P_2)|$ is a minimum. Suppose an interior vertex u exists which is adjacent to v_i and v_j, where $\{v_i, v_j\} \in C$. Although it is possible for an interior vertex to be joined to many list-2 vertices, the minimality of P_2 implies that u has no list-2 neighbours in $P_2 - \{u, v_i, v_j\}$. We also know that there are at most 2 list-2 vertices in $P_2 - \{u, v_i, v_j\}$ that are joined to v_i or v_j. We colour any such list-2 vertex. By the minimality of P_2, P_2 contains at most 3 vertices of P, and therefore, at most 5 vertices of P_2 are coloured (3 vertices of P, plus v_i and v_j's list-2 neighbours we just coloured). We first apply induction to P_1 (which gives a colour to v_i, u, and v_j), and since P_2 now has at most 6 coloured vertices (u is now coloured), the induction requirements for P_2 are satisfied, and we apply induction to P_2.

For example, in D_1 in figure 4.1, P_1 gets coloured by induction which gives colours to v_7, u, and v_3. v_1 and v_2 are already coloured since they are members of P. v_8 is a list-2 vertex which is adjacent to v_7 ($v_7 = v_j$ in this case) so v_8 receives a colour as well.

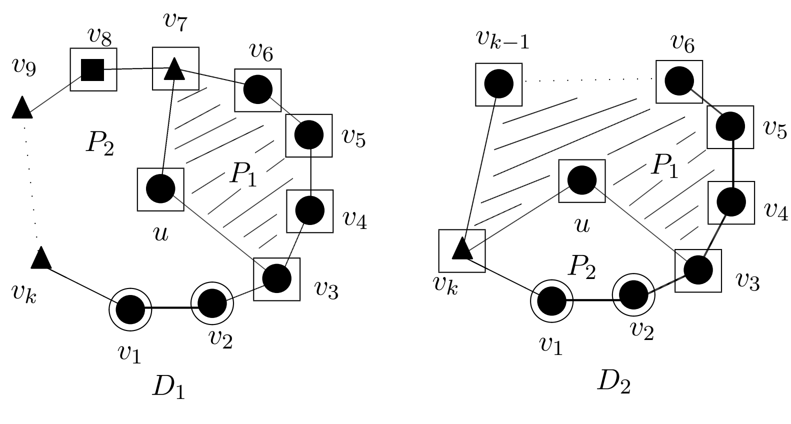

Fig. 4.1: G has a path of the form $v_j u v_i$.

Case 5 will fail if P has length 6 and the path is of the form v_3uv_k or v_4uv_7. For example, in D_2 in figure 4.1, v_k gets coloured by induction on P_1, but it is possible that it may receive the same colour as v_1 which is not allowed. Similarly, if the path containing u was of the form v_4uv_7, v_7 may receive the same colour as v_6 when it is coloured by induction on P_1.

Note that when Thomassen [10] refers to this case on page 191, he says "If $v_juv_i = v_3uv_7$, and u_8 has only two available colors ...", but he means, "... and v_8 has only two available colors ...".

By using an argument similar to Case 5 we can also conclude:

(g) G has no path of the form v_juwv_i, where u and w are in $\mathrm{int}(C)$ and v_i is a list-2 vertex.

(h) G has no path of the form v_juwv_i, where u and w are in $\mathrm{int}(C)$, v_i is a list-3 vertex, and $v_j = v_1$ or $v_j = v_q$, where v_q is the last vertex in the path P.

Case 6: v_{q+2} is a list-3 vertex.

We know that v_{q+1} is a list-3 vertex, since it is adjacent to the coloured vertex v_q. In this case, we can complete the proof by deleting v_q and deleting its colour from the list of each of its neighbours. This will create a smaller graph which allows us to use induction (figure 4.2). It is possible that this will make v_{q+1} a list-2 vertex, but we do not need to worry about it being adjacent to another list-2 vertex since C has no chords (e), v_{q+2} is list-3, and no neighbour of v_q can also be a neighbour of v_{q+1} since G has girth at least 5. We also know v_q's interior list-2 neighbours will not be adjacent to each other (G has girth at least 5), to a list-2 in $C - P$, or to a coloured vertex of P by Case 5. Since we delete v_q's colour from its neighbours' lists, v_q can be safely added back into $G - v_q$ after induction is used, and this gives us a proper 3-list-colouring of G.

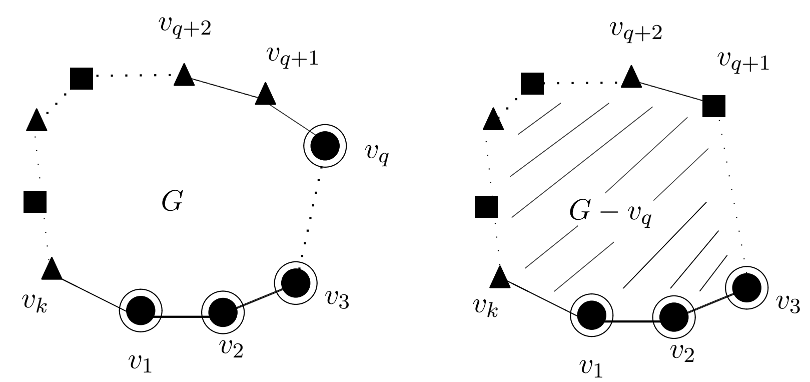

Fig. 4.2: v_{q+2} is a list-3 vertex.

Case 7: v_{q+2} is a list-2 vertex, and v_{q+4} is list-3.

Since v_{q+2} is a list-2 vertex, v_{q+3} must be list-3 since no two list-2 vertices can be adjacent. Therefore, to complete the proof in this case we first properly colour v_{q+1} and v_{q+2}, delete them, and also delete their colour from each of their respective neighbours' lists, if necessary, and apply induction to the resulting graph. A problem could arise if either v_{q+1} or v_{q+2} have an interior neighbour u which is joined to a coloured vertex, since u could now be a list-2 (recall all coloured vertices are in P). By Case 5, this is only possible if P has length 6 and G has a vertex u in int(C) joined to both v_4 and v_7. (We do not have to worry about Case 5's other exception v_3uv_k because by (d) there must be at least three vertices in the outer boundary C not in P, and therefore v_{q+1} and v_{q+2} cannot be v_k, and so v_3uv_k is not an issue since u will not be turned into a list-2 vertex in this case.)

If the v_4uv_7 case does exist, then we still colour v_{q+1} and v_{q+2}, delete them, and delete their colours from their respective neighbours' lists, and we also colour u (recall that all interior vertices of G have a list size of three, so u will have a colour in its list distinct from v_4 and v_7, and so u can be properly coloured). By (f) the interior of $v_4uv_7v_6v_5v_4$ must be empty, so we can complete the proof by applying induction to $G - \{v_5, v_6, v_7, v_8\}$, where P is now $v_1v_2v_3v_4u$ (figure 4.3).

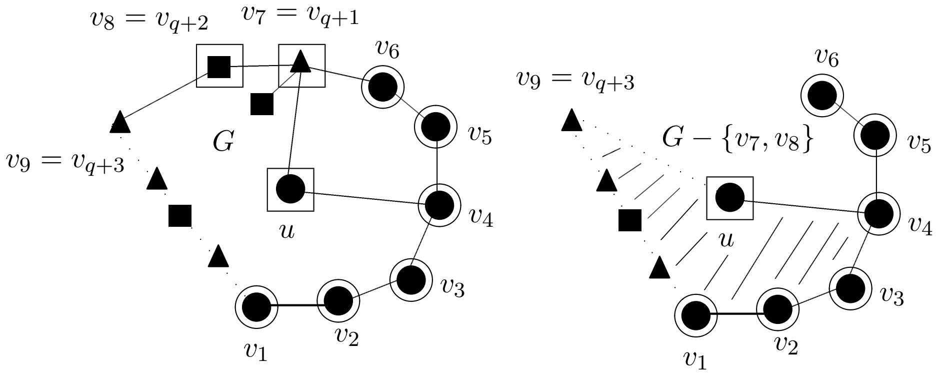

Fig. 4.3: v_{q+2} is a list-2 vertex, and v_{q+4} is list-3.

In Case 7, v_{q+3} may become a list-2 vertex when v_{q+2} deletes its colour from its neighbours' lists. This is not problematic, because since v_{q+4} is a list-3 vertex, we know that $v_{q+4} \neq v_1$ because we know v_1 is a list-1 vertex since it's coloured (in other words, $k > q + 3$). This means that $v_{q+3} \neq v_k$, and so if v_{q+3} becomes a list-2 vertex, we do not have to worry, because it will not be adjacent to the coloured vertex v_1. Case 8 handles the cases where v_{q+3} is adjacent to either a list-2 or list-1 vertex. If v_{q+3} becomes a list-2 vertex in Case 8, we would not be able to use induction in the same way as Case 7, so we must proceed differently.

Case 8: v_{q+2} is a list-2 vertex, and v_{q+4} is not list-3 . (It is possible for v_{q+4} to be list-1 when $v_{q+4} = v_1, k = q + 3$, or to be list-2.)

Since v_{q+2} is a list-2 vertex, v_{q+3} is list-3, so it will have at least one colour in its list that v_{q+4} does not. Therefore, we colour v_{q+3} using a colour not in v_{q+4}'s list, and we also properly colour v_{q+1} and v_{q+2}. We then delete these three vertices and delete their colour from the lists of their respective neighbours. We want to complete the proof by applying induction to the resulting graph, but first we must analyze the potential problems of applying induction. Case 5, (e), (g), and (h) make induction possible expect for in a few exceptional cases. Like in Case 7, we must examine what will happen if G has a v_4uv_7 or a v_3uv_k. Since v_{q+1} and v_{q+3} are both coloured in Case 8, we must also consider what will happen if they have interior neighbours z and w, respectively, which are adjacent to each other. We must examine this case because since v_{q+1} and v_{q+3} delete their colour from their neighbours' lists, this would make both w and z list-2 vertices, and we know list-2 vertices must not be adjacent in order for induction to proceed.

1. If G has an interior vertex u joined to both v_4 and v_7 then we proceed the same as in Case 7.

2. If G has an interior vertex v joined to both v_3 and v_k where $v_k = v_{q+3}$ ($v_{q+4} = v_1$), then we also colour v before we use induction, but we do not delete it. This splits G into two cycles: $P_1 = vv_3v_4v_5v_6v_7v_8v_9v$ and $P_2 = vv_3v_2v_1v_9v$. We know P_2 is empty by (f), so once we delete v_1,v_2, and v_3 we can use induction on P_1 to colour the smaller graph. (We do not need to consider this case if $v_k \neq v_{q+3}$ because v_k would not be deleted.)

3. If G has a path $v_{q+3}wzv_{q+1}$ where w and z are interior vertices, we must colour w and z, delete them, and delete their colour from their respective neighbours' lists before we apply induction (figure 4.4). Note that the cycle $E = wzv_{q+1}v_{q+2}v_{q+3}w$ is empty by (f), so the path $v_{q+3}wzv_{q+1}$ is unique.

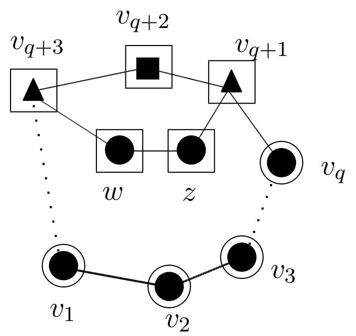

Fig. 4.4: G has a path $v_{q+3}wzv_{q+1}$.

Let w' and z' be neighbours of w and z, respectively. Since we deleted w and z's colour from their respective neighbour's lists, w' and z' may now be list-2 vertices. Therefore, before we can use induction, we must make sure w' and z' are not adjacent to any other list-2 vertices. We will take care of this possibility below.

Note that these three problems may all coexist. Our coloured vertices split our graph into parts. We want to apply induction to each part, but first we must ensure that the induction properties are met each time. We must make sure that each part has a coloured path with size at most six, and that there is no edge joining vertices with a list size less than 3 (list-2's and list-1's), except for the edges in P. If there is a vertex joined to two other coloured vertices after the additional colouring we will colour this vertex using the remaining colour from its list, but will not delete it before we apply the induction hypothesis.

Criterion (h) ensures that there are at most six pre-coloured vertices in each part. Since v_{q+1} and v_{q+3} initially have a list size of 3, the paths $v_{q+1}zz'v_1$, $v_{q+1}zz'v_6$, $v_{q+3}ww'v_1$, and $v_{q+3}ww'v_6$ cannot exist by (h).

For example, in figure 4.5 we can see that $v_{q+1}zz'v_1$ cannot exist by (h). (We must rule out $v_{q+1}zz'v_1$, because we know z' receives a colour, since it's next to two coloured vertices, so if we did not rule this case out, P would have size 7 after the additional colouring of z', and we would not be able to use induction.) Therefore, to complete the proof we colour $v_{q+1}, v_{q+2}, v_{q+3}, w$ and z, delete them, delete their colour from their respective neighbours' lists, colour z', and apply induction to the resulting graph.

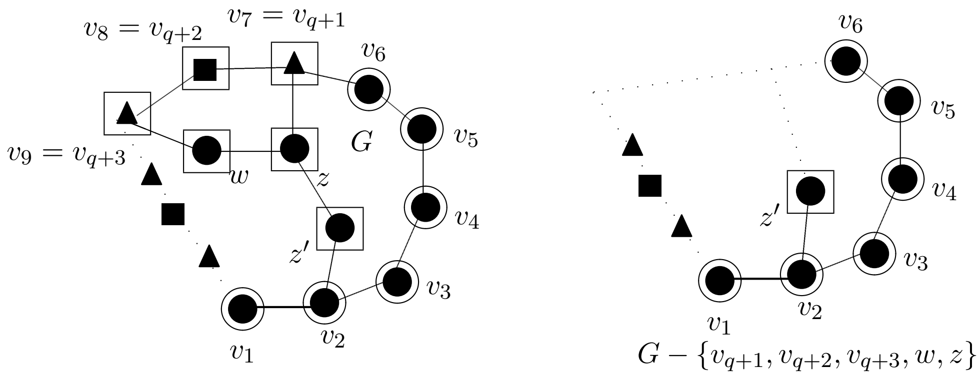

Fig. 4.5: A newly coloured vertex will not be adjacent to v_1 or v_6.

If u or v are adjacent to w' or z' this will not induce a path with size greater than six. For example, in figure 4.6 the path v_3vv_k exists, and the cycle $v_{q+3}wzv_{q+1}v_{q+2}$ also exists. $v_1, v_2, \ldots, v_6$ are already coloured, $v_{q+1}, v_{q+2}, v_{q+3}$ receive a colour, are deleted, and their colour is deleted from their neighbours' lists. v receives a colour but is not deleted. w and z receive a colour, are deleted, and their colour is deleted from their neighbours' lists. w' and z' also receive a colour, because they are adjacent to two coloured vertices after the additional colouring. Once the necessary vertices are coloured and deleted we can see that G is split into parts. The cycles $wzz'vw'w$, $wzv_7v_8v_9w$, and $vv_3v_2v_1v_kv$ are empty by (f) (see D_1 in figure 4.6). Therefore, there are two remaining parts, $v_6 \ldots v_3vz' \ldots v_6$ and $v_kvw' \ldots v_k$, each of which have at most 6 coloured vertices (see D_2 in figure 4.6). We complete the proof by applying induction to each part. The path length condition is therefore satisfied.

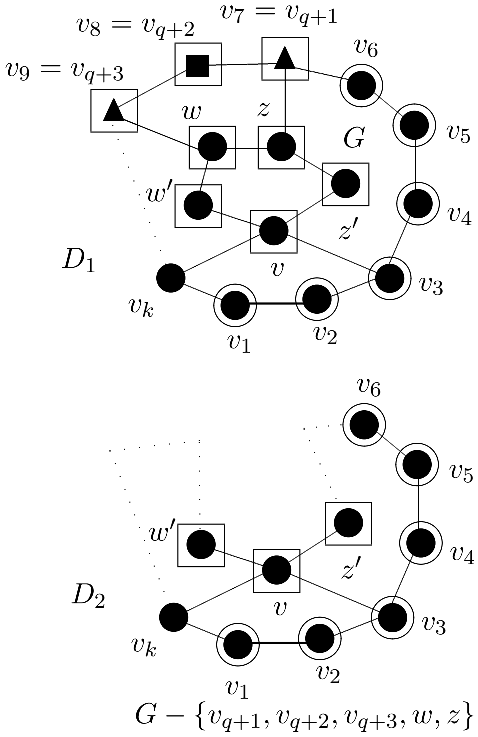

Fig. 4.6: u or v are adjacent to w' or z'.

Now we must make sure that there is no edge joining vertices with a list size less than 3 (list-2's and list-1's), except for the edges in P. We know that v_{q+1}, v_{q+2}, and v_{q+3} did not create any list-2 vertices, apart from w and z, since E is empty (figure 4.4). A list-2 vertex v_t may exist in $C - P$, but (g) and Case 5 ensure that interior list-2 vertices are not adjacent to v_t. For example, in figure 4.7 we know all interior vertices have three colours in their list. The only place a list-2 vertex could exist before any additional colouring is in $C - P$, so it is possible that v_{q+6} could be a list-2 vertex. However, there cannot be an edge between z' and v_{q+6} because (g) states that G has no path of the form $v_j uwv_i$ where u and w are in $\text{int}(C)$ and v_i is a list-2 vertex.

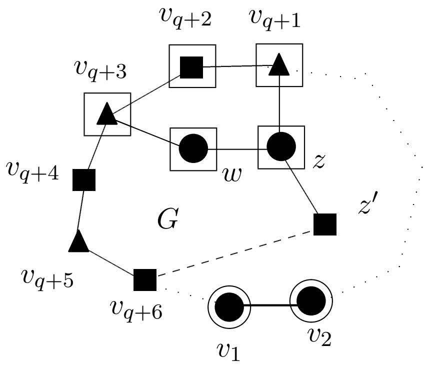

Fig. 4.7: List-2 vertices will not be adjacent.

Since G has girth 5, there is no other way that two list-2 vertices can be adjacent (figure 4.8). Therefore, w' and z' are not adjacent to any other list-2 vertices, so we can complete this case using induction. Therefore, the induction properties are satisfied, and this completes the proof.

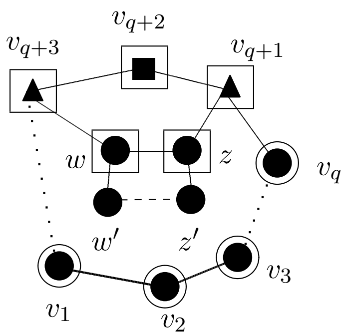

Fig. 4.8: No edge between w' and z' since G has girth 5.

Theorem 2 proved that planar graphs with girth of at least five are 3-list colourable. Since r-list-colourable is a stronger condition than r-colourable, any graph that can be r-list coloured can be r-coloured. Therefore, Theorem 2 implies that planar graphs with girth of at least five are 3-colourable. However, in order to prove Grötzsch's theorem we need to prove that all planar graphs of girth at least four (triangle-free) are three colourable. Theorem 3 will prove this result.

Theorem 3: Let G be a triangle-free planar graph. Then G can be 3-coloured.

Proof: The proof is by induction on the number of vertices in G. As in Chapter 3, let us assume that all vertices not in the outer boundary C are uncoloured, and that if C has size at most 6, then C may be uncoloured or it may be safely coloured. We can assume that G is 2-connected (not disconnected and no cut-vertex), because otherwise we could just apply induction to each part. We can also assume that every vertex not on the outer boundary C has degree at least 3, or else we could use Case 1, Chapter 3. If G has a separating 4-cycle or 5-cycle, then we can use induction (see Claim 1, Chapter 3). If we have a vertex joined to two vertices of C, then we can use induction (see Case 3, Chapter 3). If G has a face of size 4 distinct from C, then we can use induction by identifying 2 vertices in a similar way as Case 4, Chapter 3. If G has no 4-cycle then we apply Theorem 2 from Chapter 4. Therefore, since there are no triangles, no 4-faces, and no separating 4-cycles, we can assume that the only 4-cycle in G is the outer boundary $C : x_1x_2x_3x_4x_1$, since we took care of all other possible 4-cycles. If C is coloured, we know it is properly coloured, so x_1 and x_4 do not share the same colour. If C is not coloured, colour x_1 and x_4 with two different colours. Now insert an uncoloured degree 2 vertex x_5 on the edge x_4x_1. Since x_5 has degree 2 it is only adjacent to x_1 and x_4. Therefore, there will be always be a third colour left over for x_5, so we will colour x_5 with this available colour. This makes a graph with girth at least 5, so we can properly colour $G + x_5$ by using Theorem 2, Chapter 4. We know that x_1 and x_4 do not share the same colour, so when we remove x_5, G will still be properly coloured. This completes the proof.

5. CONCLUSION

In addition to simplifying Grötzsch's result, many graph theorists have also extended Grötzsch's theorem to allow for planar graphs with triangles, under certain conditions. For example, B. Grünbaum proved that any planar graph with at most 3 triangles is 3-colourable [6]. This result is best possible, since there exist planar graphs with 4 triangles which are not 3-colourable [6].

Similarly, O.V. Borodin and A. Raspaud [4] conjectured that any planar graph without 5-cycles, is three colourable as long its triangles are at a minimum distance 1 (no triangles share a common edge or vertex). If true, this result is best possible, since there exist planar graphs without 5-cycles which are not 3-colourable, and there exist planar graphs without intersecting triangles which are not three colourable [4]. Borodin and Raspaud took the first step towards proving this conjecture in 2003 when they proved that any planar graph without 5-cycles is 3-colourable, as long as its triangles are at a minimum distance 4 (at least 4 vertices away from each other) [4]. Borodin and Raspaud attained this result by determining the properties of a minimal counterexample, and then used discharging to prove that this counterexample could not exist. Borodin and Glebov [3] improved this result in 2010 when they proved any planar graph without 5-cycles is 3-colourable, as long as its triangles are at a minimum distance 2. Independently, Baogang Xu proved that planar graphs without 5-cycles and without 7-cycles are 3-colourable, as long as no triangles share a common vertex [12]. Borodin and Raspaud's conjecture still remains open.

Although the majority of graph colouring research has focused on planar graphs, Grötzsch's theorem has also been generalized to include more complex surfaces, like the torus and projective plane [11]. For example, Thomassen and Kawarabayashi proved that any planar graph of girth 5 can be decomposed into an independent set and a forest and used this result to generalize Grötzsch's theorem to allow for triangles at a minimum distance 5, as long as each triangle has a vertex v which is on the outer boundary, such that v is not contained in any 4-cycle [7]. They conjectured that their result could also be used to generalize the current 3-colour theorems for torus [11] and Klein bottle [9]. The current 3-colour proof for the torus holds for graphs without triangles and quadrilaterals [11]. However, this result could perhaps be generalized to allow for triangles at a minimum distance k.

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