λ-Harmonious Graph Colouring

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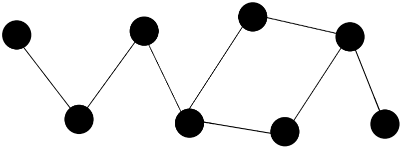
What is a graph?



● vertices

— edges

What is vertex colouring?

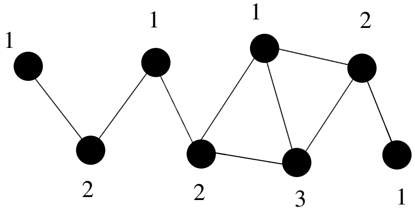


Figure : Proper Colouring.

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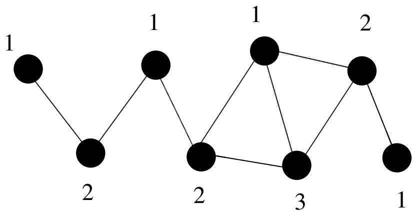

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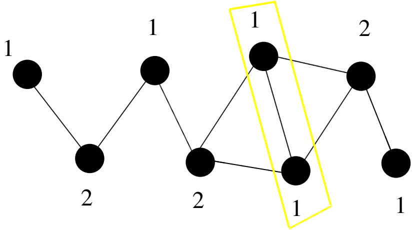

Figure : Improper Colouring.

Edge Colour Pair

- The unordered pair of colours assigned to an edge's incident vertices.

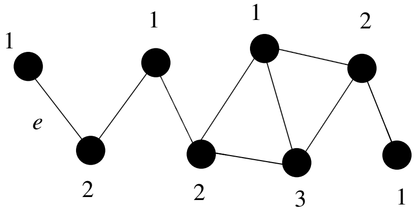

Figure : e has the colour pair $\{1,2\}$.

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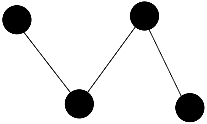

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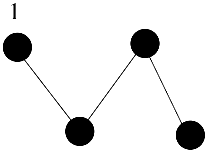

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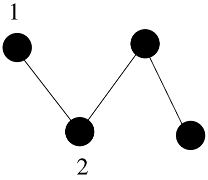

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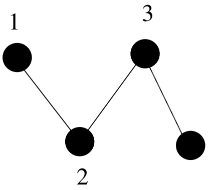

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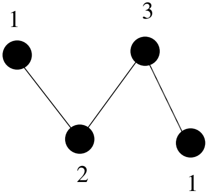

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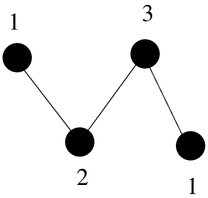

Figure : $h(P_4) = 3$.

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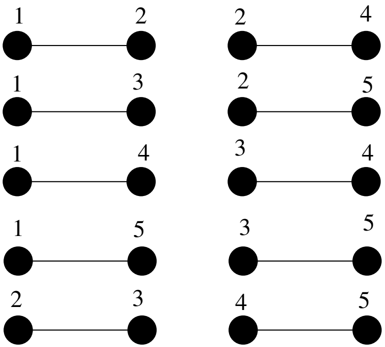

Figure : A graph with 10 edges needs at least 5 colours.

Harmonious Chromatic Number

- $h(G)$ has been found for several families of graphs like paths (Miller, Pritikin 91'), cycles (Lee, Mitchem 87'), and trees (Mitchem 89').

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- $h(G)$ has been found for several families of graphs like paths (Miller, Pritikin 91'), cycles (Lee, Mitchem 87'), and trees (Mitchem 89').
- We will generalize $h(G)$ to allow for up to λ edge colour pairs.

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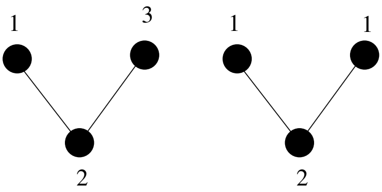

Figure : $h_1(P_3) = 3, h_2(P_3) = 2$.

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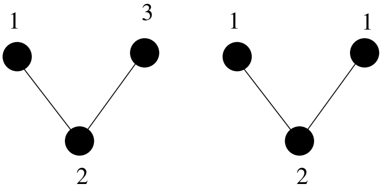

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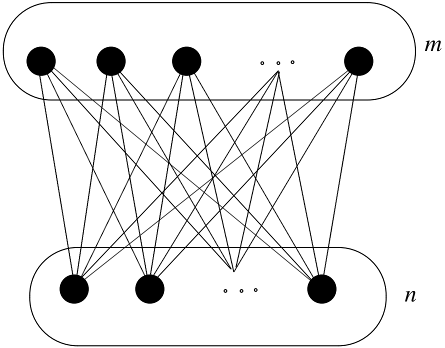
- $|E(G)| \leq \lambda \binom{k}{2}$

Complete Bipartite Graphs

- **Complete Bipartite Graph -** a graph whose vertex set can be decomposed into two disjoint sets such that no two vertices in the same set are adjacent, and every pair of vertices in distinct sets are adjacent. We will denote a complete bipartite graph by $K_{m,n}$ where m and n are the sizes of the disjoint sets, with $m \geq n$.



Complete Bipartite Graphs

- The λ-harmonious chromatic number of a complete bipartite graph $K_{m,n}$ with $m \geq n$ is:

$$\textbf{Theorem} : h_\lambda(K_{m,n}) = \left\{ min\left(\left\lceil \frac{m}{\left\lfloor \frac{\lambda}{q} \right\rfloor} \right\rceil + \left\lceil \frac{n}{q} \right\rceil ; 1 \leq q \leq \left\lfloor \sqrt{\lambda} \right\rfloor , q \in \mathbb{Z} \right) \right\}.$$

$h_4(K_{8,3})$:

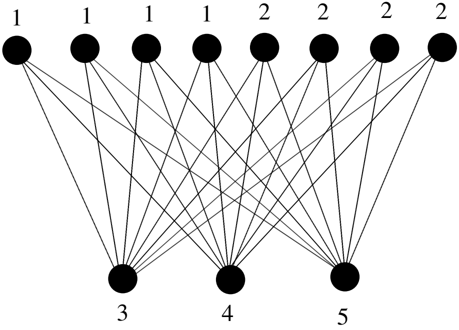


Figure : $\lceil \frac{m}{4} \rceil + n = 2 + 3 = 5$

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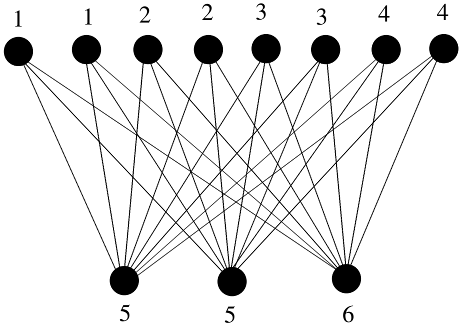

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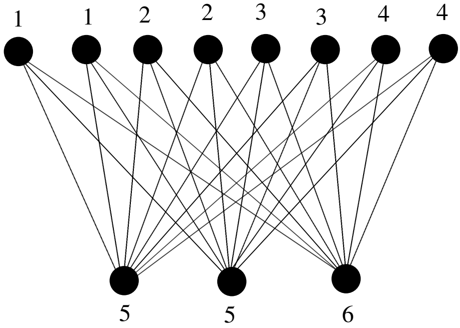

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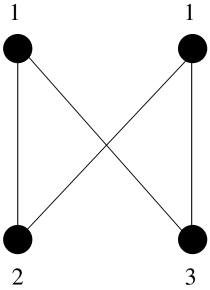
$h_4(K_{2,2})$:



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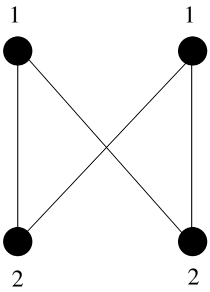

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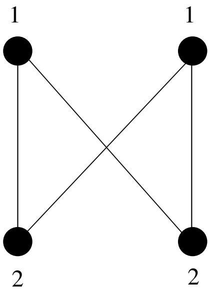

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- $h_4(K_{2,2}) = \lceil \frac{m}{2} \rceil + \lceil \frac{m}{2} \rceil = 1 + 1 = 2$

Paths

- **Path -** an alternating sequence of distinct vertices and edges that begins and ends with a vertex. We denote a path on n vertices by P_n.

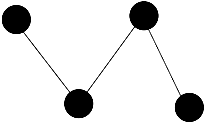

Figure : A path of length 3, denoted P_4

Paths

- The λ-harmonious chromatic number of a path, P_n, is as follows:
  Theorem: Let $r \in \mathbb{Z}$ be determined by the inequality $\lambda\binom{2r-1}{2} < n-1 \leq \lambda\binom{2r+1}{2}$.
  Then

$$h_\lambda(P_n) = \begin{cases} 2r & \text{if } \lambda \text{ is even and } n-1 \leq \lambda\binom{2r}{2}, \\ & \text{or, if } \lambda \text{ is odd and } n-1 \leq \lambda\binom{2r}{2} - (r-1), \\ 2r+1 & \text{otherwise.} \end{cases}$$

Proof Idea for Paths:

- An **Eulerian path** is a trail in a graph that visits each edge exactly once.

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- A graph has an Eulerian path $\Leftrightarrow$ it is connected and has at most two vertices of odd degree.
- A **complete graph** on n vertices, K_n, is a graph such that each pair of vertices is connected by an edge.

Proof Idea for Paths ($\lambda = 1$):

- Properly colour a complete graph K_n (i.e. give each vertex a unique colour).

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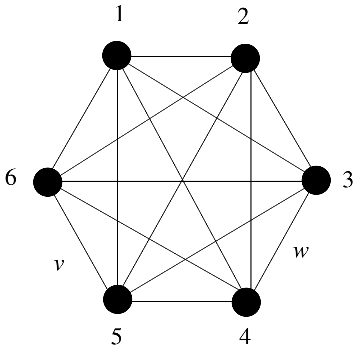
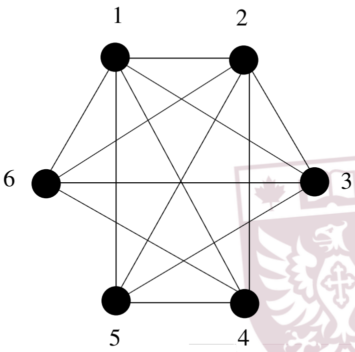
- Properly colour a complete graph K_n (i.e. give each vertex a unique colour).
- If there exists an Eulerian path of length k in K_n, then this gives us a proper colouring of the path P_{k+1}.
- Given a path on m vertices, P_m, the harmonious chromatic number, $h_1(P_m)$ will be the smallest k such that there exists a complete graph K_k with an Eulerian path of length $m-1$.

Example:

- We know $\binom{5}{2} < 11, 12, 13, 14, 15 \leq \binom{6}{2}$
  $\Rightarrow h_1(P_{12}), h_1(P_{13}), h_1(P_{14}), h_1(P_{15}), h_1(P_{16}) \geq 6.$
- So, we consider K_6.
- Each edge has odd degree, so we must delete at least 2 edges to have an Eulerian path. After removing v and w we're left with only two vertices of odd degree, so we can trace an Eulerian path 13524614512632, and we can see $h_1(P_{12}) = h_1(P_{13}) = h_1(P_{14}) = 6$.





Proof Idea for Paths ($\lambda = n$):

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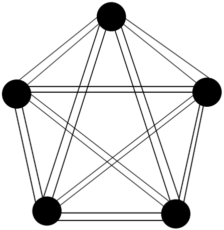

Figure : $H_{2,5}$.

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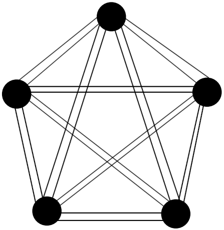

Figure : $H_{2,5}$.

- Given a path on m vertices, P_m, the λ-harmonious chromatic number, $h_\lambda(P_m)$, will be the smallest k such that $H_{\lambda,k}$ has a subgraph with an eulerian path of length $m-1$.

Cycles

- **Cycle -** a closed path where the first and last vertex are the same. We denote a cycle on n vertices by C_n.

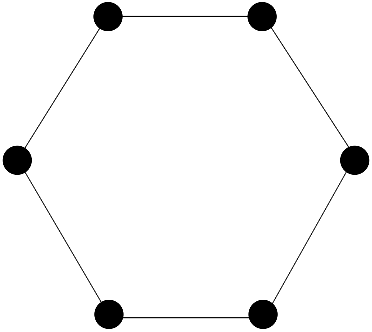

Figure : A cycle of length 6, denoted C_6

Cycles

- The λ-harmonious chromatic number of a cycle, C_n, is as follows:
  Theorem: Let k be the least integer such that $n \le \lambda \binom{k}{2}$. Then

$$h_\lambda(C_n) = \begin{cases} k & \text{if one of the following four conditions hold:} \\ & \quad i)\ \lambda \text{ is even and } n \neq \lambda\binom{k}{2} - 1, \\ & \quad ii)\ \lambda \neq 1, \lambda \text{ is odd, } k \text{ is odd, and } n \neq \lambda\binom{k}{2} - 1, \\ & \quad ii)\ \lambda = 1, k \text{ is odd, and } n \neq \lambda\binom{k}{2} - i \text{ for } i = 1,2, \\ & \quad iv)\ \lambda \text{ is odd, } k \text{ is even, and } n \neq \lambda\binom{k}{2} - i \text{ for } i = 0 \ldots \frac{k}{2} - 1, \\ k+1 & \text{otherwise.} \end{cases},$$

Example:

- If we have a cycle C_7, then we know we $h_1(C_7) \geq 5$, since $\binom{4}{2} < 7 < \binom{5}{2}$. So, we consider K_5. If we delete the edges from a cycle of length 3 in K_5, then we are left with a subgraph of K_5 with an eulerian cycle of length 7. Therefore, $h(C_7) = 5$.

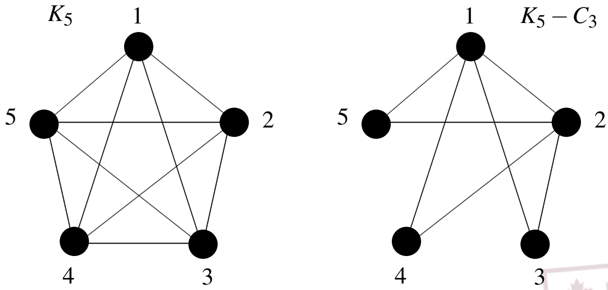

Figure : An eulerian subgraph of K_5 with 7 edges.

Wheels

- **Wheel -** a graph on n vertices formed by connecting a single vertex to all vertices of a C_{n-1}. We denote a wheel on n vertices as W_n.

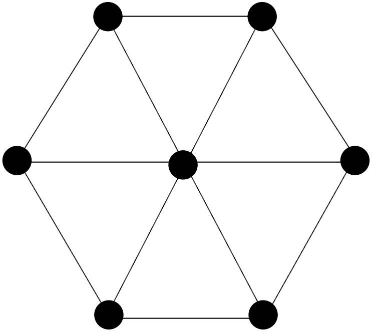

Figure : A wheel on 7 vertices, denoted W_7

Wheels

- Note that each wheel contains a star $K_{n-1,1}$ and a cycle C_{n-1}.

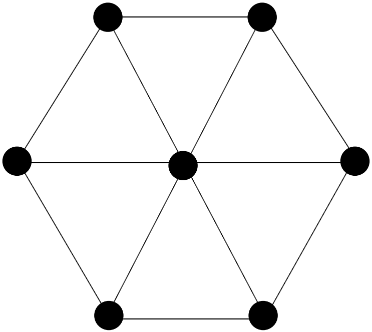

Figure : W_7

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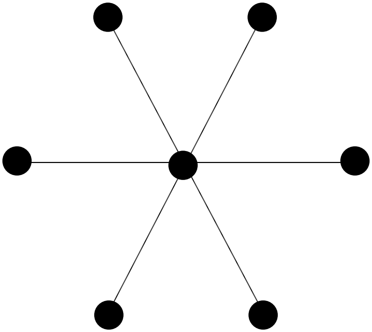

Figure : W_7 contains a star $K_{6,1}$

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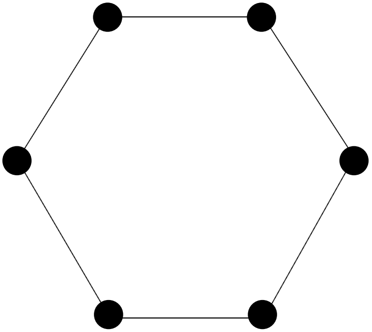

Figure : W_7 contains a cycle C_6

Wheels

- **Theorem:** Let $h_\lambda(K_{n-1,1}) = \left\lceil \frac{n-1}{\lambda} \right\rceil + 1 = t$. If $t > 4$, then $h_\lambda(W_n) = t$.

Future Work

- Although $h(G)$ has been found for several families of graphs, the harmonious colouring problem has been proven to be NP-complete (Johnson 83').
    - i.e. Given a graph G and a positive integer $k \leq |V(G)|$, can G be harmoniously coloured with k colours?
- We suspect that the λ-harmonious colouring problem is also NP-complete, but we haven't been able to prove it yet.
    - i.e. Given a graph G and a positive integer $k \leq |V(G)|$, can G be λ-harmoniously coloured with k colours?

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Thank you