4.7: Row Space, Column Space, & Null Space

Let A be an $m \times n$ matrix.

<u>Recall</u>:
- The <u>row space</u> of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.
- The <u>column space</u> of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
- The <u>nullspace</u> of A is the subspace of $\mathbb{R}^n$ spanned by the solutions to the homogeneous system $Ax = 0$.

4.8: Rank - Nullity

Let A be an $m \times n$ matrix.

Theorem 4.8.1 (pg. 337)

<u>Recall</u>:
- $\text{rank}(A) = \dim\{\text{column space of } A\}$. $= \dim\{\text{rowspace}\}$.
- $\text{nullity}(A) = \dim\{\text{nullspace of } A\}$.
- $\text{rank}(A) + \text{nullity}(A) = n$.

A <u>Example</u>: Let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$.

i.e. 7 @ most 3 leading 1's in r.r.e.f.

1. (a) The rank of A is at most $\underline{\ 3\ }$.

   (b) The nullity of A is at most $\underline{\ 4\ }$.

   i.e. 7 # free parameters in $Ax=0$ is at most 4.

   (c) The rank of A^T is at most $\underline{\ 3\ }$.

   (d) The nullity of A^T is at most $\underline{\ 3\ }$.

$\boxed{2}$ Find a basis for the nullspace of A.

Want to find the solution space of $Ax = 0$.

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right] \begin{array}{l} \\ r_2 \leftarrow r_2 - 2r_1 \\ r_3 \leftarrow r_3 + r_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right] r_3 \leftarrow r_3 + r_2$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] r_2 \leftarrow r_2 \cdot -\tfrac{1}{7}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -4x_2 - 5x_3 - 2x_4$

$x_2 = -x_3 - 4/7\, x_4 = -t - 4/7\, s$

$x_3 = t$

$x_4 = s$.

$x_1 = -4(-x_3 - \tfrac{4}{7}x_4) - 5x_3 - 2x_4$

$= 4x_3 + 16/7\, x_4 - 5x_3 - 2x_4$

$= -x_3 + 2/7\, x_4$

$= -t + 2/7\, s$.

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix} s$$

is the solution space for $Ax=0$.

$$\left\{ \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}}_{v_2} \right\}$$ is a basis for the nullspace of A.

i.e. v_1 & v_2 are linearly independent & $\text{span}\{v_1, v_2\} = \text{nullspace}(A)$.

3 What is nullity(A)?

nullity(A) = dim{nullspace A} = 2.

4 What is rank(A)?

$\text{rank}(A) = \underbrace{4}_{\#\text{ columns}} - \underbrace{2}_{\text{nullity}(A)} = 2.$ by the Rank-Nullity Theorem

5 Find a basis for the rowspace of A.

We already put A in row-echelon form:

$$\begin{bmatrix} \boxed{1} & 4 & 5 & 2 \\ 0 & \boxed{1} & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So,

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 4/7 \end{bmatrix} \right\}$$

is a basis for the rowspace of A.

6 Find a basis for the columnspace of A.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

The leading 1's in the row echelon form of A tell us which columns to use. (see Theorem 4.76, pg. 231).

$\boxed{7}$ Find a basis for the rowspace of A consisting of original rows of A.

We should row reduce A^T, & the leading 1's point to which rows we should use.

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \begin{matrix} r_2 \leftarrow r_2 - 4r_1 \\ r_3 \leftarrow r_3 - 5r_1 \\ r_4 \leftarrow r_4 - 2r_1 \end{matrix} \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 7 \\ 0 & -7 & 7 \\ 0 & -4 & 4 \end{bmatrix} \begin{matrix} r_3 \leftarrow r_3 - r_2 \\ r_4 \leftarrow r_4 - \frac{4}{7}r_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} r_2 \leftarrow r_2 \cdot -\tfrac{1}{7} \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So our basis is

$$\mathcal{R} := \left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}.$$

- Recall: Elementary row operations do not change dependence relations among column vectors. i.e. If $w_1, \ldots, w_k$ are linearly dependent columns in A (i.e. $c_1w_1 + \cdots + c_kw_k = 0$) & if $w_1', \ldots, w_k'$ are what the columns of A become after row op's, then $c_1w_1' + \cdots + c_kw_k' = 0$. The converse is true as well. (pg. 230).

[to verify, think about how row op's ⟷ elementary matrices.]

$\boxed{8}$ Write the third row of A as a linear combination of the other two. (i.e. Find the coordinate vector of $[-1,3,2,2]$ w.r.t. the basis R).

Easy to see by looking at the row echelon form of A^T:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 \\ 3 \\ 2 \\ 2 \end{bmatrix} = -\begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}.$$

$\boxed{9}$ Find a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

i.e. Find a basis for the column space of the matrix $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$. (see #$\boxed{6}$).

$\boxed{B}$ Suppose A is a 3x3 matrix whose nullspace is a line through the origin in $\mathbb{R}^3$.

(4.8, #11).

Can the row or column space of A be a line through the origin too?

No! By the rank-nullity Theorem,

$$\dim\{\cancel{\text{row}}\ \text{columnspace}\} = \underbrace{3}_{\#\text{columns}} - \underbrace{1}_{\text{nullity}(A)} = 2$$

$\Rightarrow$ the column space of A is a plane.

Recall: $\dim\{\text{column space } A\} = \dim\{\text{row space } A\}$.

So $\dim\{\text{row space } A\} = 2$.

not a line... lines are 1-dim'l.

$\boxed{C}$ Suppose A is a 3x5 matrix.

$\boxed{1}$ Are the row vectors of A linearly dependent?

$\boxed{2}$ Are the column vectors of A linearly dependent?

$$A = 3\underbrace{\begin{bmatrix} & & & & \end{bmatrix}}_{5}.$$

$\dim\{\text{column space } A\} \le 3$

b/c $\dim\{\text{row space}\} \le 3$ b/c only 3 rows $\Rightarrow$ the column vectors must be linearly dependent.

Not enough info to make a conclusion about the row vectors.

$\boxed{D}$ In ⓐ-ⓖ, is $Ax=b$ consistent? If so, state the # of parameters in its general solution.

4.8 #7

	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ	ⓕ	ⓖ
Size of A	3x3	3x3	3x3	5x9	5x9	4x4	6x2
rank (A)	3	2	1	2	2	0	2
rank[A|b]	3	3	1	2	3	0	2

Recall: $\overset{A_{n\times n}}{A}x=b$ consistent (for any b) $\Leftrightarrow$ rank $A = n$ $\Leftrightarrow$ $Ax=b$ has exactly one solution.

ⓐ A 3x3 & rank(A) = 3 ⇒ $Ax=b$ consistent for any b. ✓

rank(A) = 3 & A 3x3 ⇒ $Ax=b$ has exactly one solution ⇒ # parameters = 0.

ⓑ rank(A) = 2, but rank[A|b] = 3 ⇒ in the r.r.e.f. of A have a row of 0's, but in r.r.e.f. A|b have no row of zeros

⇒ looks like $\begin{bmatrix} 1 & 0 & 0 & \vdots & * \\ 0 & 1 & 0 & \vdots & * \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$ ⇒ not consistent.

ⓒ rank A = rank(A|b) ⇒ don't get a "1" in one of A's zero rows (when A in r.r.e.f.) ⇒ consistent.

parameters = nullity(A) = 3 − 1 = 2.

Hilroy

(d) $\text{rank}(A) = \text{rank}(A|b) = 2 \Rightarrow$ consistent for same reason as last time.

$\#\text{parameters} = 9 - 2 = 7.$

(e) $\underbrace{\text{rank}(A|b)}_{=3} > \underbrace{\text{rank}(A)}_{=2} \Rightarrow$ not consistent.

(f) $\text{rank}A = \text{rank}(A|b) \Rightarrow$ consistent.

$\#\text{parameters} = 4 - 0 = 4.$

(g) $\text{rank}(A) = \underbrace{\text{rank}(A|b)}_{=2} \Rightarrow$ consistent.

$\#\text{parameters} = 2 - 2 = 0.$