2013 = blue

2014 = green

Extra Problems = purple

Math 2203 - Test #2 Review Session

mult. by x

3.4: Undetermined Coefficients [#1, 3, 4] [#1, ~~1~~] [#13] easy

3.5: Variation of Parameters [#2, 5a] [#9] [#1, 3]

3.6: Cauchy - Euler [#5] [# ~~1~~, 12] [#1, 14]

3.8: Linear Models: IVP's (Spring/Mass Systems) [#4] [#3, 4] [#4]

3.9: Linear Models: BVP's (Beams & EigenFunctions) [#3] [#2] [#6, 12]

quickly undet coef

Boundary conditions

eigen function

8.8: Eigenvalues [#7, 8b] [#10] [#8, 15]

8.9: Cayley - Hamilton Theorem [#8] [#7]

8.10: Orthogonal / Symmetric Matrices [#5]

8.12: Diagonalization [#6]

I had some requests from students who couldn't make the review to post my review session notes. Most of the questions we did were straight from the two practice tests which are posted on avenue, so I'm not sure how helpful any of the following is... but I'll post it anyways in case anyone want to see it!

(1) Solution $x^2y'' + xy' = 1$ $y(1) = 1$ $y'(1) = -1/2$ $x>0$

We try $y = x^m$ to find the complementary solution

This produces the auxiliary equation

$$m(m-1) + m = 0$$

$$m^2 - m + m = 0 \Rightarrow m^2 = 0 \Rightarrow m = 0 \text{ and } 0$$

$\therefore$ the two independent homogeneous solutions are

$$y_1 = x^0 = 1 \qquad y_2 = x^0 \ln x = \ln x$$

Variation of Parameters:

$$y_c = c_1 1 + c_2 \ln x = c_1 \underbrace{1}_{y_1} + c_2 \underbrace{\ln x}_{y_2}$$

$$W(y_1, y_2) = \begin{vmatrix} 1 & \ln x \\ 0 & 1/x \end{vmatrix} = 1/x \neq 0 \text{ on } x>0$$

To find y_p, we use variation of parameters

First we must put the equation in standard form.

$$y'' + \frac{1}{x} y' = \underbrace{\frac{1}{x^2}}_{f(x)}$$

Then

$$u_1' = \frac{\begin{vmatrix} 0 & \ln x \\ 1/x^2 & 1/x \end{vmatrix}}{} = \frac{-\ln x}{x^2} = \frac{-\ln x}{x^2}$$

$$u_1 = -\int \frac{\ln x}{x} dx = -\frac{(\ln x)^2}{2} + \cancel{c}^{\,0}$$

$u = \ln x$, $du = \frac{1}{x}dx$, $\int u\, du$ — using substitution

$$u_2' = \begin{vmatrix} 1 & 0 \\ 0 & 1/x^2 \end{vmatrix} = \frac{1}{x^2} \qquad u_2 = \int \frac{1}{x} dx = \ln x + \cancel{c}^{\,0}$$

$$\therefore y = c_1 + c_2 \ln x - \frac{(\ln x)^2}{2} \cdot 1 + (\ln x)(\ln x)$$

$$= c_1 + c_2 \ln x + (\ln x)^2/2$$

Apply I.C. $y(1) = c_1 + 0 + 0 = 1 \Rightarrow c_1 = 1$

$$y' = 0 + \frac{c_2}{x} + \frac{\ln x}{x} \qquad y'(1) = c_2 = -1/2 \qquad \therefore y = 1 - \frac{1}{2}\ln x + \frac{(\ln x)^2}{2}$$

NAME: ____________________ ID #:____________________

Part I: Provide all details and fully justify your answer in order to receive credit.

#12

(*12 pts.*) Find all of the eigenvalues and the corresponding eigenfunctions for the boundary value problem,

Eigenfunctions

$$y''(t) + (3+\lambda)\,y(t) = 0,\ y'(0) = 0, y'(\pi) = 0.$$

Make sure to consider all the cases.

Solution. The auxiliary equation is $m^2 + (3+\lambda) = 0$. There are 3 cases to consider: $3+\lambda < 0$, $3+\lambda = 0$ and $3+\lambda > 0$.

$m = \pm\sqrt{-3-\lambda}$

Case 1: $3+\lambda < 0$:
We can let $3+\lambda = -\omega^2$ where $\omega > 0$. The auxiliary equation becomes $m^2 - \omega^2 = 0$ and has roots $m = \pm\omega$. The general solution is then

$$y = C_1\, e^{\omega x} + C_2\, e^{-\omega x} \quad \text{and} \quad y' = C_1\,\omega\, e^{\omega x} - C_2\,\omega\, e^{-\omega x}.$$

The condition $y'(0) = 0$ yields $C_1\,\omega - C_2\,\omega = 0$ or $C_1 = C_2$. The condition $y'(\pi) = 0$ yields $C_1\,\omega\,(e^{\omega\pi} - e^{-\omega\pi}) = 0$ and thus $C_1 = 0$ since $\omega > 0$ and $e^{\omega\pi} - e^{-\omega\pi} > 0$. Thus no non-trivial solution exists in this case.

Case 2: $3+\lambda = 0$:
The auxiliary equation becomes $m^2 = 0$ and $m = 0$ is a double root. The general solution is

$$y = C_1 + C_2\,x \quad \text{and} \quad y' = C_2.$$

The condition $y'(0) = 0$ yields $C_2 = 0$ and thus the other condition $y'(\pi) = 0$ automatically holds. The function $\boxed{\phi_0(x) = 1}$ is thus an eigenfunction corresponding to the eigenvalue $\boxed{\lambda_0 = -3}$.

Case 3: $3+\lambda > 0$:
We can let $3+\lambda = \omega^2$ where $\omega > 0$. The auxiliary equation becomes $m^2 + \omega^2 = 0$ and has roots $m = \pm i\,\omega$.
The general solution is then

$$y = C_1\,\cos(\omega x) + C_2\,\sin(\omega x) \quad \text{and} \quad y' = -C_1\,\omega\,\sin(\omega x) + C_2\,\omega\,\cos(\omega x).$$

The condition $y'(0) = 0$ yields $C_2\,\omega = 0$ or $C_2 = 0$. Thus, $y' = -C_1\,\sin(\omega x)$. The condition $y'(\pi) = 0$ implies that $C_1\,\sin(\omega\pi) = 0$. For non-trivial solutions to exist, we need $\sin(\omega\pi) = 0$ which can only happen if $\omega = k$, where $k \geq 1$ is an integer. This yields the eigenfunctions $\boxed{\phi_k(x) = \cos(kx)}$ corresponding to the eigenvalues $\boxed{\lambda_k = k^2 - 3, k \geq 1}$,

Continued...

8.

a) [3 marks] The eigen~~vectors~~ values of 2x2 matrix A satisfy the characteristic polynomial

$$\lambda^2 - 3\lambda + 2 = 0$$

Cayley-Hamilton

$\cancel{(\lambda-2)(\lambda-1)=0 \quad \lambda=2, \lambda=1.}$

Determine A if

$$A^2 = \begin{bmatrix} -14 & 6 \\ -45 & 19 \end{bmatrix}$$

$A^2 - 3A + 2I = 0$

$\frac{1}{3}A^2 + \frac{2}{3}I = A.$

$$\frac{1}{3}A^2 + \frac{2}{3}I = \begin{bmatrix} -\frac{14}{3} + \frac{2}{3} & 2 \\ -15 & \frac{19}{3} + \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 \\ -15 & 7 \end{bmatrix}.$$

b) [2marks] You are given that

Eigenvectors

$$X = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \quad \text{is an eigenvector of} \quad A = \begin{bmatrix} -8 & 2 & 16 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

But you are only given the entries for the first row of matrix A. Knowing what it means for X to be an eigenvector of A, what value is the associated eigenvalue of X?

$AX = \lambda X \Rightarrow -16 + 10 + 16 = \lambda 2 \Rightarrow 10 = 2\lambda \Rightarrow \lambda = 5.$

END OF TEST QUESTIONS

Undetermined Coef.

SOLUTIONS GROUP A

1. (3 marks) Which of the following could be a particular solution to the following non-homogeneous equation

$$y'' - y = e^x \ ?$$

By undetermined coef. has form:

(a) $y = \frac{1}{2}(e^x + e^{-x})$

(b) $y = \frac{1}{2}xe^x$

(c) $y = \frac{1}{2}xe^{2x}$

(d) $y = \frac{1}{2}e^{2x}$

(e) $y = e^x - 1$

$y_p = Axe^x$. only choice is $\boxed{b}$.

$y_p' = Ae^x + Axe^x$

$y_p'' = 2Ae^x + Axe^x$.

$y_p'' - y_p = e^x \iff 2Ae^x = e^x \Rightarrow A = 1/2$.

$\therefore \boxed{y_p = \frac{1}{2}xe^x}$.

- homogeneous $y'' - y = 0 \Rightarrow m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y_{Hom}(x) = c_1 e^x + c_2 e^{-x}$

Beams

2. (3 marks) A flagpole is being blown by the wind. The base of the flagpole is at $x = 0$, while the top of the pole is at $x = x_b$. If

$$EI\frac{d^4y}{dx^4} = w(x)$$

is an ODE for the deflection of the pole from vertical, with E and I constants, what are the boundary conditions?

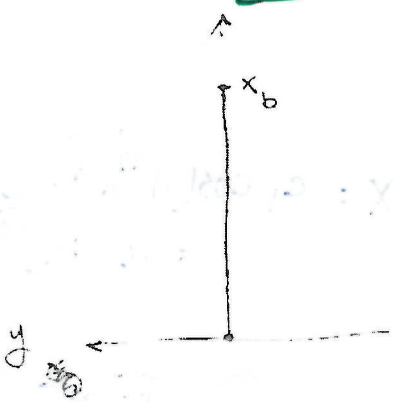
(a) $y(0) = 0,\quad y''(0) = 0\quad y'(x_b) = 0\quad y''(x_b) = 0$

(b) $y(0) = 0,\quad y''(0) = 0\quad y(x_b) = 0\quad y'(x_b) = 0$

(c) $y'(0) = 0,\quad y''(0) = 0\quad y(x_b) = 0\quad y''(x_b) = 0$

(d) $y'(0) = 0,\quad y'''(0) = 0\quad y''(x_b) = 0\quad y'''(x_b) = 0$

(e) $y(0) = 0,\quad y'(0) = 0\quad y''(x_b) = 0\quad y'''(x_b) = 0$



embedded $\left\{ \begin{array}{l} y(0) = 0 \quad (x=0 \text{ is fixed}) \\ y'(0) = 0 \quad (\text{vertical tangent at } x=0) \end{array} \right.$

Free $\left\{ \begin{array}{l} y''(x_b) = 0 \quad (\text{bending moment} = 0) \\ y'''(x_b) = 0 \quad (\text{shear force} = 0) \end{array} \right.$

Springs

3. (3 marks) The displacement of a mass on a spring is given by $x(t)$, where the spring is at rest when $x(t) = 0$. The motion of the mass is given by Newton's Law of motion:

$$F = ma$$

The forces acting on the mass are a spring force of $-kx$, and a "driving force" of $3\sin(4t)$. We will assume there is no friction. If $k = 20$, for what mass, m, do we get resonance, creating and oscillation whose amplitude grows linearly in time, t?

(a) 4

$\boxed{\text{(b)}}$ 5/4

(c) 12

(d) 2

(e) 3/2

$$m x'' + \underbrace{20}_{K} x = \underbrace{3\sin(4t)}_{F(t)}$$

$$\Downarrow$$

$$x'' + \frac{20}{m} x = \frac{3}{m}\sin 4t$$

$$\lambda^2 + \frac{20}{m} = 0 \rightarrow \lambda = \pm i\sqrt{\frac{20}{m}}$$

$$X = c_1 \cos\left(\sqrt{\tfrac{20}{m}}\, x\right) + c_2 \sin\left(\sqrt{\tfrac{20}{m}}\, x\right).$$

See pg. 158/159. $\Rightarrow$ resonance for $4 = \sqrt{\frac{20}{m}} \Rightarrow 16 = \frac{20}{m}$

$\Rightarrow m = \frac{20}{16} = \boxed{\frac{5}{4}}$.

If $\sqrt{\frac{20}{m}} \neq 4$, then

$x_p = A\sin(4t) + B\sin(4t)$

If $\sqrt{\frac{20}{m}} = 4$

then

$x_p = t\left[A\sin(4t) + B\sin(4t)\right]$

Diagonalization

6. (3 marks) Find a 2×2 matrix that has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$, and corresponding eigenvectors $v_1 = [1, 2]^T$ and $v_2 = [1, 1]^T$.

(a) $\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

$\boxed{\text{(d)}}$ $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix}$

$$M = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad M^{-1} = -\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$$

$$M^{-1} A M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} =: D$$

$$A = M D M^{-1}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$