Math 2203 - Review [Midterm #1]

1st-Order DE's

- **1.1 & 1.2: Theory / Terminology**
  - Existence / Uniqueness
- **2.2: Separable Eq^{n}'s**
  - 2.1: Autonomous DE's
    - critical pts
    - stable/unstable
    - Logistic Eq^{n} [2.8]
  - Newton's Law of Heating / Cooling
- **2.3: Linear Eq^{n}'s**
  - solving via integrating Factor
  - Mixture of 2 salt solutions [2.7]
- **Geometric & Numerical Analysis**
  - 2.1: Direction Fields
  - 2.6: Euler's Method

Higher Order DE's

- **3.1: Theory of Linear DE's**
  - Existence/Uniqueness
  - Linear Independence
  - Wronskian
  - Fundamental set of solutions
  - General solution [of homog. & nonhomog. DE]
- **3.3: Homog. Linear w/ constant Coef.'s**
  - Find roots to aux. eq^{n} & solve

Analyzing Solutions of DE's:

Solve Symbolically	Qualitatively	Geometrically	Numerically
• Separable • 1st-order Linear • Homog. Linear w/ Constant coef.	• Existence / Uniqueness • General solution of Linear DE's	• Direction Fields • Autonomous DE's	• Euler's Method

NAME: ______________________ ID #:______________________

5. *(4 pts.)* Consider the following differential equation:

$$y^{(4)}(t) - 7\,y'''(t) + 19\,y''(t) - 13\,y'(t) = 0$$

Linear Homog. Constant Coef.

Then general solution is given by:

→ **(A)** $c_1 + c_2\,e^{t} + c_3\,e^{3t}\cos(2t) + c_4\,e^{3t}\sin(2t)$, $c_i \cancel{=} 0$, $i = 1, 2, 3, 4$, arbitrary.

(B) $c_1 + c_2\,e^{t} + c_3\,e^{2t}\cos(3t) + c_4\,e^{2t}\sin(3t)$, $c_i = 0$, $i = 1, 2, 3, 4$, arbitrary.

(C) $c_1 + c_2 e^{t} + c_3\,e^{2t}\cos(4t) + c_4\,e^{2t}\sin(4t)$, $c_i = 0$, $i = 1, 2, 3, 4$, arbitrary.

(D) $c_1 + c_2\,e^{3t} + c_3\,e^{3t}\cos(4t) + c_4\,e^{3t}\sin(4t)$, $c_i = 0$, $i = 1, 2, 3, 4$, arbitrary.

(E) $c_1\,t + c_2\,e^{2t} + c_3\,e^{4t}\cos(3t) + c_4\,e^{4t}\sin(3t)$, $c_i = 0$, $i = 1, 2, 3, 4$, arbitrary.

Solution. The auxiliary equation

$$m^4 - 7\,m^3 + 19\,m^2 - 13\,m = 0$$

has the the obvious roots $m = 0$ and $m = 1$. It can be factorized as

$$m\,(m-1)(m^2 - 6\,m + 13) = 0 \quad \text{or} \quad m\,(m-1)\,(m-3)^2 + 2^2) = 0$$

$$\begin{array}{r} 4\,52 \\ 36 \\ \hline 16 \end{array} \qquad \begin{array}{r} 13 \\ 4 \\ \hline 52 \end{array}$$

The roots are thus $m = 0, 1, 3 \pm 2\,i$ (all simple). The general solution has thus the form

$$y(t) = c_1 + c_2\,e^{t} + c_3\,e^{3t}\cos(2t) + c_4\,e^{3t}\sin(2t),$$

where c_i, $i = 1, 2, 3, 4$, are arbitrary.

$m(m-1)(m^2 - 6m + 13)$

$m^4 - 7m^3 + 19m^2 - 13m = 0$

$\dfrac{6 \pm \sqrt{36 - 4(13)}}{2}$

$m[m^3 - 7m^2 + 19m - 13] = 0.$

$m = 1$ root.

$= 3 \pm 2i.$

$$\begin{array}{r} m^2 - 6m + 13 \\ m - 1\,\big)\overline{m^3 - 7m^2 + 19m - 13} \\ m^3 - m^2 \\ \hline -6m^2 + 19m - 13 \\ -6m^2 + 6m \\ \hline 13m - 13 \\ 13m - 13 \\ \hline 0 \end{array}$$

$\therefore y = c_1 + c_2 e^{t} + e^{3t}[c_3\cos(2t) + c_4\sin(2t)$

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NAME: ____________________ ID #:____________________

2. *(4 pts.)* Find the general solution of the differential equation

linear homog.
constant coef.

$$y^{(6)} - 2\,y^{(3)} + y = 0.$$

The answer (where $C_1, \ldots, C_6$ denote arbitrary constants) is:

(a) $y(x) = C_1\,e^{x} + C_2\,e^{-x} + C_3\,\cos(x) + C_4\,\sin(x) + C_5\,x\,\cos(x) + C_6\,x\,\sin(x)$.

(b) $y(x) = C_1\,e^{x} + C_2\,x\,e^{x} + C_3\,e^{-x/2}\cos(\frac{\sqrt{3}}{2}\,x) + C_4\,e^{-x/2}\sin(\frac{\sqrt{3}}{2}\,x) + C_5\,x\,e^{-x/2}\cos(\frac{\sqrt{3}}{2}\,x) + C_6\,x\,e^{-x/2}\sin(\frac{\sqrt{3}}{2}\,x)$

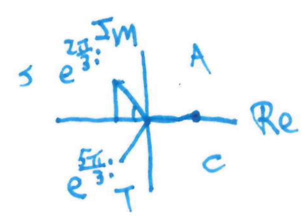
(c) $y(x) = C_1\,e^{x} + C_2\,x\,e^{x} + C_3\,e^{-x/2}\cos(\frac{\sqrt{3}}{2}\,x) + C_4\,e^{-x/2}\sin(\frac{\sqrt{3}}{2}\,x) + C_5\,e^{x/2}\cos(\frac{\sqrt{3}}{2}\,x) + C_6\,e^{x/2}\sin(\frac{\sqrt{3}}{2}\,x)$

(d) $y(x) = C_1\,e^{x} + C_2\,x\,e^{x} + C_3\,e^{-x/2}\cos(2\,x) + C_4\,e^{-x/2}\sin(2\,x) + C_5\,x\,e^{-x/2}\cos(2\,x) + C_6\,x\,e^{-x/2}\sin(2\,x)$

(e) $y(x) = C_1\,e^{x} + C_2\,x\,e^{x} + C_3\,e^{-x} + C_4\,x\,e^{-x} + C_5\,e^{-x/2}\cos(\frac{\sqrt{3}}{2}\,x) + C_6\,e^{-x/2}\sin(\frac{\sqrt{3}}{2}\,x)$

$m^6 - 2m^3 + 1 = 0$ $\quad m = 1$ root. $\quad (m^3-1)^2 = m^6 - 2m^3 + 1.$

$m^3 = 1$



$1 = e^{0i}.$ $\quad \frac{2\pi}{3}$

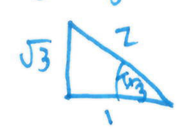
$e^{\frac{2\pi}{3}i} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})$
$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$

$\boxed{\therefore\ y = c_1 e^{x} + c_2 x e^{x} + e^{-\frac{x}{2}}\left[c_3\cos(x\tfrac{\sqrt{3}}{2}) + c_4\sin(\tfrac{\sqrt{3}}{2}x)\right] + x e^{-\frac{x}{2}}\left[c_5\cos(\tfrac{\sqrt{3}}{2}x) + c_6\sin(\tfrac{\sqrt{3}}{2}x)\right]}$

$m = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$

double roots.

$\pi - \frac{2\pi}{3} = \frac{\pi}{3}$



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4. Determine whether the Existence/Uniqueness Theorem guarantees that the DE $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point.

$\boxed{a}$ $(1, 4)$, $\boxed{b}$ $(2, -3)$.

Here $F(x,y) = \sqrt{y^2 - 9}$. The Theorem is satisfied for points inside (on the interior) of a region R, where F & $\frac{\partial F}{\partial y}$ are cont. on R.

$F(x,y) = \sqrt{y^2 - 9}$ is cont. so long as $y^2 - 9 \geq 0$

$\Leftrightarrow y^2 \geq 9 \Leftrightarrow |y| \geq 3 \Leftrightarrow y \geq 3$ or $y \leq 3$.

$\frac{\partial F}{\partial y} = \frac{y}{\sqrt{y^2-9}}$ is cont $\Leftrightarrow y^2 - 9 > 0$

$\Leftrightarrow y > 3$ or $y < -3$.

So, we could choose any points (x,y) s.t. y lies in the interval $(-\infty, -3)$ or $(3, \infty)$.

$\boxed{a}$ $4 \in (3, \infty)$, so Theorem satisfied for $(1, 4)$. ✓

$\boxed{b}$ $-3 \notin (-\infty, -3)$ & $-3 \notin (3, \infty)$, so Theorem not satisfied for $(2, -3)$.

7. (3 marks) Consider the following Boundary Value Problem

$$\begin{cases} y'' = ky \\ y(0) = y(2\pi) = 0 \end{cases}$$

For what value of k does the BVP admit a non-trivial solution?

(a) $k = 1$
(b) $k = -9$
(c) $k = 0$
(d) $k = \pi$
(e) $k = 4$

$y'' = Ky$

$y'' - Ky = 0$

$m^2 - K = 0$

$m^2 = K$

$m = \pm\sqrt{K}$.

$y = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$ if $K \neq 0$ &

$y = \underbrace{c_1 + xc_2}_{0 = y(0) = y(2\pi) \Rightarrow c_1 = c_2 = 0.}$ if $K = 0$.

$0 = c_1 + c_2$

$0 = c_1 e^{\sqrt{K}(2\pi)} - c_1 e^{-\sqrt{K}(2\pi)}$

$= \underbrace{c_1 \left[e^{\sqrt{K}(2\pi)} - e^{-\sqrt{K}(2\pi)} \right]}_{\text{assuming } K \neq 0 \text{ here, so } K \text{ must be negative.}}$.

$K = -9$: $m^2 = -9$

$m = \pm 3i$.

$y = c_1 \cos(3x) + c_2 \sin(3x)$.

$0 = c_1$. c_2 can be anything.

$0 = c_1$.

NAME: ____________________ ID #:____________________

Part II: Provide all details and fully justify your answer in order to receive credit.

$u = 1+x^2$
$du = 2x\,dx$

6. *(10 pts.)* Compute in explicit form the solution of the initial value problem

$$\begin{cases} (1+x^2)\,y' - 2\,x\,y = x\,(1+x^2)^2\,e^x, \\ y(0) = 1. \end{cases}$$ Linear.

$\boxed{\text{Linear}}$

$\int P(x)dx = \int \frac{-2x}{1+x^2}\,dx$

$= \int \frac{-1}{u}\,du = -\ln(1+x^2).$

$y' - \frac{2x}{1+x^2}y = \frac{x(1+x^2)^2 e^x}{1+x^2}$

$y' - \underbrace{\frac{2x}{1+x^2}}_{P(x)} y = \underbrace{xe^x(1+x^2)}_{f(x)}$

$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} f(x)dx\right] = (1+x^2)\left[\int \frac{1}{1+x^2} xe^x(1+x^2)\,dx\right]$

$= (1+x^2)\left[\int xe^x dx\right] = (1+x^2)\left[xe^x - \int e^x dx\right]$

$u = x \quad v = e^x$
$du = dx \quad dv = e^x$

$= (1+x^2)\left[xe^x - e^x + c\right].$ $y(0)=1$

$\Rightarrow 1 = [-e^0 + c] \Rightarrow c = 2.$

$\therefore \boxed{y = (1+x^2)\left[xe^x - e^x + 2\right]}.$

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