Math 2203 - Tutorial #9

1. Consider $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$. Find A^K using the Cayley-Hamilton Theorem.

$$0 = \begin{vmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda) \Rightarrow \lambda = -1,\ \lambda = -3.$$

Characteristic Eq'n: $\lambda^2 + 4\lambda + 3 = 0.$

By Cayley-Hamilton Theorem, $A^2 + 4A + 3I = 0$, so it's always possible to write $A^K = c_0 I + c_1 A$ for some constants c_1 & c_2, & $\lambda^K = c_0 + c_1 \lambda$.

$$\lambda^K = c_0 + c_1\lambda \Rightarrow \left.\begin{aligned} (-3)^K &= c_0 + c_1(-3) \\ (-1)^K &= c_0 + c_1(-1) \end{aligned}\right\} \quad \begin{aligned} c_0 - 3c_1 &= (-3)^K \\ c_0 - c_1 &= (-1)^K \end{aligned}$$

$$-2c_1 = (-3)^K - (-1)^K$$

$$c_1 = -\tfrac{1}{2}(-3)^K + \tfrac{1}{2}(-1)^K.$$

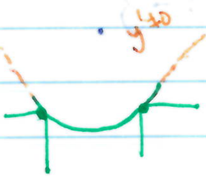
$$c_0 = (-1)^K + c_1 \Rightarrow c_0 = -\tfrac{1}{2}(-3)^K + \tfrac{3}{2}(-1)^K.$$

$$\text{So, } A^K = c_0 I + c_1 A = \begin{bmatrix} -\frac{1}{2}(-3)^K + \frac{3}{2}(-1)^K & 0 \\ 0 & -\frac{1}{2}(-3)^K + \frac{3}{2}(-1)^K \end{bmatrix} + c_1 \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^K & -(-3)^K + (-1)^K \\ 0 & (-3)^K \end{bmatrix}.$$

$\{ y''$ bending moment
$\{ y'''$ shear force

2. Suppose a shopkeeper wants to put up a rectangular sign of length L for his store, & that the deflection of the sign can be modelled by the 4th-order DE $EI\, y^{(4)} = w(x)$. Identify the appropriate boundary conditions:

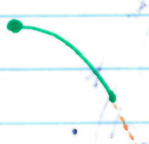


[a] He uses one nail on each side.

Hinged on both sides: $y(0)=0$, $y''(0)=0$; $y(L)=0$, $y''(L)=0$.

[b] He uses 2 nails on each side.

Embedded on both sides. $y(0)=0$, $y'(0)=0$; $y(L)=0$, $y'(L)=0$.

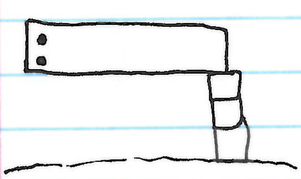


[c] He use 3 nails on the left & no nails on the right.

Embedded on left & free on right.

$y(0)=0$ $y''(L)=0$
$y'(0)=0$ $y'''(L)=0$

[d] He use 2 nails on left & a stack of crates on right.



Embedded on left & simply supported on right.

$y(0)=0$ $y(L)=0$
$y'(0)=0$ $y''(L)=0$.

3. Find the deflection, $y(x)$, in $\boxed{d}$ if $w(x) = w_0$ a constant & $0 < x < L$.

$$y^{(4)} = \frac{w_0}{EI}, \quad y(0)=0, \; y'(0)=0, \; y(L)=0, \; y''(L)=0.$$

$m^4 = 0$, $m = 0$ w/ multiplicity 4.

$$y_c = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

Undetermined Coef.:

$$y_p = Ax^4 \qquad y_p^{(4)} = \frac{w_0}{EI} \iff 24A = \frac{w_0}{EI} \iff A = \frac{w_0}{24EI}.$$

$$y'_p = 4Ax^3$$

$$y''_p = 12Ax^2$$

$$y'''_p = 24Ax$$

$$y^{(4)}_p = 24A.$$

$$\therefore \; y = y_c + y_p = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{w_0}{24EI} x^4.$$

$$y(0) = 0 \Rightarrow 0 = c_1 \qquad y(L) = 0 \Rightarrow c_3 L^2 + c_4 L^3 + \frac{w_0}{24EI} L^4 = 0$$

$$y'(0) = 0 \Rightarrow 0 = c_2.$$

$$y''(L) = 0 \Rightarrow 2c_3 + 6c_4 L + \frac{w_0}{2EI} L^2 = 0$$

$$y_p = c_3 x^2 + c_4 x^3 + \left(\frac{w_0}{24EI}\right) x^4.$$

$$y'_p = 2c_3 x + 3c_4 x^2 + \frac{w_0}{6EI} x^3$$

$$y''_p = 2c_3 + 6c_4 x + \frac{w_0}{2EI} x^2.$$

$$c_3 = -c_4 L - \frac{w_0}{24EI} L^2.$$

$$c_4 = -\frac{w_0}{12EI} L - \frac{2c_3}{3L} = -\frac{w_0}{12EI} L + \frac{c_4}{3} + \frac{w_0}{24EI} \frac{L}{3}$$

$$\Rightarrow \frac{2}{3} c_4 = \frac{-5w_0}{72EI} L \Rightarrow c_4 = \frac{-5w_0}{48EI} L \Rightarrow c_3 = \frac{5w_0}{48EI} L^2 - \frac{w_0}{24EI} L^2 = \frac{w_0}{16EI} L^2.$$

$$\therefore \; \boxed{y = \frac{w_0}{16EI} L^2 x^2 - \frac{5}{48} \frac{w_0}{EI} L x^3 + \frac{w_0}{24EI} x^4 = \frac{w_0}{48EI}\left[3L^2x^2 - 5Lx^3 + 2x^4\right].}$$

4. $\boxed{a}$ Solve the homog. system of linear DE's:

$$X' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

$$0 = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 24 - 10\lambda + \lambda^2 + 5 = \lambda^2 - 10\lambda + 29.$$

$\begin{array}{r} {}^{3}29 \\ 4 \\ \hline 116 \end{array}$

$$\lambda = \frac{10 \pm \sqrt{100 - 116}}{2} = \underbrace{5}_{\alpha} \pm \underbrace{2}_{\beta} i.$$

$\underline{\lambda = 5 + 2i:}$

$$\left[\begin{array}{cc|c} 6-5-2i & -1 & 0 \\ 5 & 4-5-2i & 0 \end{array}\right] \therefore \left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right] r_2 \leftarrow r_2 + (-1-2i) r_1$$

$$\left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad y = (1-2i)x = (1-2i)t, \quad x = t \quad \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B_1} + \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{B_2} i.$$

So, $X_1 = e^{5t}\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin(2t)\right]$

& $X_2 = e^{5t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos(2t)\right).$

$X = c_1 X_1 + c_2 X_2.$

$$X(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 1 & -2 & 8 \end{array}\right] r_2 \leftarrow r_2 - r_1 \quad \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & -2 & 10 \end{array}\right] \quad \begin{array}{l} c_1 = -2 \\ -2c_2 = 10 \Rightarrow c_2 = -5. \end{array}$$

$$\therefore X = -2e^{5t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin(2t)\right) - 5e^{5t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos(2t)\right)$$

$$= \begin{bmatrix} -2e^{5t}\cos(2t) - 5e^{5t}\sin(2t) \\ -2e^{5t}\cos(2t) - 2e^{5t}\sin(2t) - 5e^{5t}\sin(2t) + 10e^{5t}\cos(2t) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{5t}\left(2\cos(2t) + 5\sin(2t)\right) \\ e^{5t}\left(8\cos(2t) - 9\sin(2t)\right) \end{bmatrix}.$$

$\boxed{b}$ Sketch the solution curve corresponding to this IVP.

Complex eigenvalues $\underbrace{5}_{\alpha} \pm \underbrace{2}_{\beta} i$.

<u>$\alpha > 0$:</u> We'll have a spiral moving away from origin, passing through $(x(0), y(0)) = (-2, 8)$.