Math 2203 - Tutorial #8

1. Find the eigenvalues & eigenfunctions for the BVP
$y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$.

$m^2 + \lambda = 0 \iff m^2 = -\lambda \iff m = \pm\sqrt{-\lambda}$.

- **If $-\lambda > 0$:** $\Rightarrow \pm\sqrt{-\lambda}$ distinct real roots

$\Rightarrow y = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$.

$y(0) = 0 \Rightarrow 0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$y(\pi) = 0 \Rightarrow 0 = c_1 e^{\sqrt{-\lambda}\,\pi} + c_2 e^{-\sqrt{-\lambda}\,\pi}$

$c_2 = -c_1$

$\Rightarrow c_1\left[e^{\sqrt{-\lambda}\,\pi} - e^{-\sqrt{-\lambda}\,\pi}\right] = 0 \Rightarrow c_1 e^{\pi}\Big[\underbrace{e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}}_{\text{Not zero b/c } \sqrt{-\lambda} \neq -\sqrt{-\lambda}}\Big] = 0$

$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0$

$\Rightarrow y = 0$ only solution.

- **If $\lambda = 0$:** Solution has form $y = c_1 + c_2 x$.

$y(0) = 0 \Rightarrow c_1 = 0$. $y(\pi) = 0 \Rightarrow 0 = 0 + c_2(\pi) \Rightarrow c_2 = 0$

$\Rightarrow y = 0$ only solution.

- **If $-\lambda < 0$:** $\Rightarrow \overset{\lambda > 0 \Rightarrow}{\pm\sqrt{-\lambda}}$ complex roots $\sqrt{\lambda}\, i$

$\Rightarrow y = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$.

$y(0) = 0 \Rightarrow 0 = c_1$. $y(\pi) = 0 \Rightarrow 0 = c_2 \sin(\sqrt{\lambda}\,\pi)$

$\Rightarrow \sqrt{\lambda} \in \mathbb{Z} \Rightarrow \sqrt{\lambda} = n$ for some $n \in \mathbb{Z} \Rightarrow \lambda = n^2$ for $n = 1, 2, 3, \ldots$

$\therefore$ The eigenfunctions are $y = \sin(nx)$
& the eigenvalues are $\lambda = n^2$ for $n = 1, 2, 3, \ldots$

2. Consider $A = \begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}$.   (a) What are the eigenvalues of A?

<u>Recall</u>: • If A is square, then $\vec{x} \in \mathbb{R}^n$ s.t. $\vec{x} \neq \vec{0}$ is called an <u>eigenvector</u> of A if $\boxed{A\vec{x} = \lambda \vec{x}}$ for some $\lambda \in \mathbb{R}$. (i.e., $A\vec{x}$ is a scalar multiple of $\vec{x}$). The scalar λ is called an <u>eigenvalue</u> of A, & $\vec{x}$ is λ's corresponding eigenvector.

• $A\vec{x} = \lambda\vec{x} \Leftrightarrow A\vec{x} - \lambda\vec{x} = 0 \Leftrightarrow (A - \lambda I)\vec{x} = 0$.

by our list of equivalent statements: We know $\det(A) = 0 \Leftrightarrow A$ not invertible $\Leftrightarrow A\vec{x} = 0$ has non-trivial solutions.

So, since we're looking for vectors $\vec{x}$ s.t. $(A - \lambda I)\vec{x} = 0$, & we know that $\vec{x} \neq 0$ by def'n, then by our equivalent statements, that must mean that $\det(A - \lambda I) = 0$.

So, λ is an <u>eigenvalue</u> of A $\Leftrightarrow$ it satisfies the eq'n $\boxed{\det(A - \lambda I) = 0.}$ called the <u>characteristic eq'n</u>

So, $\det(A - \lambda I) = \begin{vmatrix} 8-\lambda & 9 \\ -6 & -7-\lambda \end{vmatrix} = (8-\lambda)(-7-\lambda) + 54$

$= -56 - 8\lambda + 7\lambda + \lambda^2 + 54 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$

$\Leftrightarrow \lambda = 2$ or $\lambda = -1$.

So, the eigenvalues of A are 2 & -1.

ⓑ Find all eigenvectors of A.

We know $\vec{x} \neq 0$ is an eigenvector if $A\vec{x} = \lambda \vec{x}$ for some $\lambda \in \mathbb{R}$, or equivalently, if $(A - \lambda I)\vec{x} = 0$. We already know that $\lambda = 2$ & $\lambda = -1$ are eigenvalues.

$\underline{\lambda = -1}$: $0 = (A - \lambda I)\vec{x} = \left(\begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\vec{x} = \begin{bmatrix} 9 & 9 \\ -6 & -6 \end{bmatrix}\vec{x} = 0.$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. So, we want to solve:

$$\begin{bmatrix} 9 & 9 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 9 & 9 & 0 \\ -6 & -6 & 0 \end{array}\right] \begin{matrix} r_1 \leftarrow r_1 * \frac{1}{9} \\ r_2 \leftarrow r_2 * -\frac{1}{6} \end{matrix} \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] r_2 \leftarrow r_2 - r_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} x = -y = -t \\ y = t. \end{matrix}$$

So, $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ is a solution to this eq'n. So, the eigenvectors of A corresponding to $\lambda = -1$ are $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ for any $t \in \mathbb{R}$.

(We say $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a "basis" for the eigenspace corresponding to $\lambda = -1$.) (don't worry about this yet).

$\underline{\lambda = 2}$: Similarly: $\begin{bmatrix} 8-\lambda & 9 \\ -6 & -7-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 8-2 & 9 & 0 \\ -6 & -7-2 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 6 & 9 & 0 \\ -6 & -9 & 0 \end{array}\right] r_2 \leftarrow r_2 + r_1 \quad \left[\begin{array}{cc|c} 6 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$6x = -9y \Rightarrow x = \frac{-9}{6}t = \frac{-3}{2}t$

$y = t$

So, $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} t$ are the set of eigenvectors corresponding to $\lambda = 2$. $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 2$.

don't worry about what this means yet!

<u>Note</u>: Since $t \in \mathbb{R}$, we know if $t = 2$, then $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} * 2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is an eigenvector.

$\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} t$ gives us an eigenvector for any value of t.

Do our answers satisfy $A\vec{x} = \lambda\vec{x}$?

<u>Check</u>

$$\overbrace{\begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}^{\vec{x}}$$

using any eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ works here e.g. $\begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -16+18 \\ 12-14 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -1 * \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ ✓

$$= \begin{bmatrix} -8 + 9 \\ 6 - 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underbrace{-1}_{\lambda} * \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}}. \checkmark$$

$$\underbrace{\begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} -24 + 18 \\ 18 - 14 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix} = \underbrace{2}_{\lambda} \underbrace{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}_{\vec{x}}. \checkmark$$

3. Consider $A = \begin{bmatrix} 5 & -3 \\ a & b \end{bmatrix}$ and suppose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What must the eigenvalue λ corresponding to $\vec{x}$ be?

By def^{n} $\vec{x} \neq 0$ is an eigenvector of A if

$$\boxed{A\vec{x} = \lambda\vec{x}} \quad \text{for some } \lambda \in \mathbb{R}.$$

$$A\vec{x} = \begin{bmatrix} 5 & -3 \\ a & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ a+b \end{bmatrix} = \underbrace{2}_{\lambda} \underbrace{\begin{bmatrix} 1 \\ \frac{1}{2}(a+b) \end{bmatrix}}_{\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

Since we know $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector, we need to get that "1" in the top row. We need to factor out a 2 to do this, so $\lambda = 2$ must be the eigenvalue. (It doesn't matter what a & b are ... looking at the first row suffices)

7. Consider $B = \begin{bmatrix}1 & 1 & -2\\ -1 & 2 & 1\\ 0 & 1 & -1\end{bmatrix}$. Find B^{-1} using the Cayley-Hamilton Theorem.

Cayley-Hamilton Theorem: An $n \times n$ matrix satisfies its own characteristic eqn!

$$0 = \begin{vmatrix}1-\lambda & 1 & -2\\ -1 & 2-\lambda & 1\\ 0 & 1 & -1-\lambda\end{vmatrix} \iff (1-\lambda)\begin{vmatrix}2-\lambda & 1\\ 1 & -1-\lambda\end{vmatrix} + \begin{vmatrix}1 & -2\\ 1 & -1-\lambda\end{vmatrix} = 0$$

$$\iff (1-\lambda)\left[(2-\lambda)(-1-\lambda) - 1\right] + -1-\lambda+2 = 0$$

$$\iff (1-\lambda)\left[-2-\lambda+\lambda^2-1+1\right] = 0$$

$$\iff (1-\lambda)(\lambda^2-\lambda-2) = 0 \iff \lambda^2-\lambda-2-\lambda^3+\lambda^2+2\lambda = 0$$

$$\iff -\lambda^3+2\lambda^2+\lambda-2 = 0.$$

$*B^{-1}$

$$\therefore -B^3+2B^2+B-2 = 0 \Rightarrow -B^2+2B+I-2B^{-1} = 0$$

$$\Rightarrow B^{-1} = -\tfrac{1}{2}B^2 + B + \tfrac{1}{2}I = -\tfrac{1}{2}\begin{bmatrix}1 & 1 & -2\\ -1 & 2 & 1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & -2\\ -1 & 2 & 1\\ 0 & 1 & -1\end{bmatrix} + B + \tfrac{1}{2}I$$

$$= -\tfrac{1}{2}\begin{bmatrix}0 & 1 & 1\\ -3 & 4 & 3\\ -1 & 1 & 2\end{bmatrix} + \begin{bmatrix}3/2 & 1 & -2\\ -1 & 5/2 & 1\\ 0 & 1 & -1/2\end{bmatrix} = \begin{bmatrix}3/2 & 1/2 & -5/2\\ 1/2 & 1/2 & -1/2\\ 1/2 & 1/2 & -3/2\end{bmatrix}.$$

5. $\boxed{a}$ Is the matrix $C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ orthogonal? $\boxed{b}$ Symmetric?

<u>Recall</u>: An $n \times n$ nonsingular matrix A is <u>orthogonal</u> if $A^{-1} = A^T$. i.e., if $A^T A = I$.

$$C^T C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{orthogonal.}$$

<u>Recall</u>: An $n \times n$ matrix A is <u>symmetric</u> if $A = A^T$.

Here $C = C^T \Rightarrow$ symmetric.

$\boxed{c}$ Does C have real eigenvalues.

Yes, b/c symmetric matrices w/ real entries have real eigenvalues.

6. Consider $A = \begin{bmatrix} -2 & -27 & 9 \\ 0 & -2 & 0 \\ 0 & -18 & 4 \end{bmatrix}$. Find A^k.

$$\begin{vmatrix} -2-\lambda & -27 & 9 \\ 0 & -2-\lambda & 0 \\ 0 & -18 & 4-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & 0 \\ -18 & 4-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-2-\lambda)(4-\lambda)$$

$$= -(2+\lambda)(-1)(2+\lambda)(4-\lambda)$$

So, $\lambda_1 = -2$ & $\lambda_2 = 4$ are eigenvalues.

$\underline{\lambda_1 = -2}$:
$$\left[\begin{array}{ccc:c} -2+2 & -27 & 9 & 0 \\ 0 & -2+2 & 0 & 0 \\ 0 & -18 & 4+2 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc:c} 0 & -27 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -18 & 6 & 0 \end{array}\right]$$

$r_1 \leftarrow r_1 * \frac{1}{9}$
$r_3 \leftarrow r_3 * \frac{1}{6}$

$$\left[\begin{array}{ccc:c} 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right] \quad r_1 \leftarrow r_1 - r_3 \qquad \left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right]$$

$3y = z$
$\Rightarrow y = \frac{1}{3}z = \frac{1}{3}s$
$x = t$
$z = s$

So, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix} s$ solves this system of eq^{ln}s $\Rightarrow$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ are the eigenvectors corresponding to $\lambda_1 = -2$.

(could have said $\begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix}$ here... same thing).

$\underline{\lambda_2 = 4}$:
$$\left[\begin{array}{ccc:c} -2-4 & -27 & 9 & 0 \\ 0 & -2-4 & 0 & 0 \\ 0 & -18 & 4-4 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc:c} -6 & -27 & 9 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & -18 & 0 & 0 \end{array}\right]$$

$r_1 \leftarrow r_1 * \frac{1}{3}$
$r_2 \leftarrow r_2 * \frac{1}{6}$
$r_3 \leftarrow r_3 * -\frac{1}{18}$

$$\left[\begin{array}{ccc:c} -2 & -9 & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r_1 \leftarrow r_1 - 9r_2 \\ r_3 \leftarrow r_3 + r_2 \end{array} \qquad \begin{bmatrix} -2 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$y = 0$
$2x = 3z$
$\Rightarrow x = \frac{3}{2}z = \frac{3}{2}t$
$z = t$

So, $\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} t$ solves this system of eq^{ln}s $\Rightarrow$ $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = 4$.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 - 3r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 \leftarrow r_3 * \frac{1}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - 3r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{9}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

So, $P^{-1} = \begin{bmatrix} 1 & \frac{9}{2} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

Check. $PP^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{9}{2} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

So, $A = PDP^{-1}$

$\Rightarrow A^K = P D^K P^{-1}$

$\Rightarrow A^K = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} (-2)^K & 0 & 0 \\ 0 & (-2)^K & 0 \\ 0 & 0 & 4^K \end{bmatrix} \begin{bmatrix} 1 & \frac{9}{2} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.