Math 2203 – Tutorial #7

1. Solve the linear system of eq$^{\text{n}}$'s:

$$\begin{cases} 3x + 2y + z = 1 \\ 5x + 4y + 2z = -1. \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 5 & 4 & 2 & -1 \end{array}\right] r_2 \leftarrow r_2 - 2r_1$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ -1 & 0 & 0 & -3 \end{array}\right] r_1 \leftarrow r_1 + 3r_2$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & -8 \\ -1 & 0 & 0 & -3 \end{array}\right] \begin{array}{l} r_1 \leftarrow r_1 \cdot \frac{1}{2} \\ r_2 \leftarrow r_2 \cdot -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & \frac{1}{2} & -4 \\ 1 & 0 & 0 & 3 \end{array}\right] r_1 \leftrightarrow r_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & \frac{1}{2} & -4 \end{array}\right] \quad \text{r.r.e.f.}$$

$x = 3$

$y = -4 - \frac{1}{2}z = -4 - \frac{1}{2}t.$

$z = t$

\# free variables = # columns − # leading 1's
= 3 − 2 = 1.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} t}_{\text{homogenous solutions}}.$$

2. Find the inverse of $A = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix}$ using row operations.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 6 & 7 & 5 & 0 & 1 & 0 \\ 3 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} r_2 \leftarrow r_2 - 6r_1 \\ r_3 \leftarrow r_3 - 3r_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -6 & 1 & 0 \\ 0 & -1 & 0 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} r_1 \leftarrow r_1 - r_2 \\ r_3 \leftarrow r_3 + r_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 7 & -1 & 0 \\ 0 & 1 & -1 & -6 & 1 & 0 \\ 0 & 0 & -1 & -9 & 1 & 1 \end{array}\right) r_3 \leftarrow r_3 * -1 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 7 & -1 & 0 \\ 0 & 1 & -1 & -6 & 1 & 0 \\ 0 & 0 & 1 & 9 & -1 & -1 \end{array}\right)$$

$$\begin{array}{l} r_2 \leftarrow r_2 + r_3 \\ r_1 \leftarrow r_1 - 2r_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 1 & 2 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 9 & -1 & -1 \end{array}\right).$$

So, $A^{-1} = \begin{pmatrix} -11 & 1 & 2 \\ 3 & 0 & -1 \\ 9 & -1 & -1 \end{pmatrix}$.

Check

$$\begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} -11 & 1 & 2 \\ 3 & 0 & -1 \\ 9 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

3. Consider $A = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{pmatrix}$. Find A^{-1} using the adjoint method.

<u>Recall</u>: If A is invertible, then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$,

where $\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$.

cofactors

$$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} -3 & 1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} -1 & -3 \\ -2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ -2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ -1 & -3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (6+1) & -(2+2) & (1-6) \\ -(-4+1) & (0+2) & -(0+4) \\ (2+3) & -(0+1) & (0+2) \end{bmatrix}^T = \begin{bmatrix} 7 & -4 & -5 \\ 3 & 2 & -4 \\ 5 & -1 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & 3 & 5 \\ -4 & 2 & -1 \\ -5 & -4 & 2 \end{bmatrix}.$$

Notice $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$\to AA^{-1} = \frac{1}{\det(A)} A \operatorname{adj}(A)$

$\to I * \det(A) = A \operatorname{adj}(A)$.

$$\det(A) * I = A * \operatorname{adj}(A) = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 7 & 3 & 5 \\ -4 & 2 & -1 \\ -5 & -4 & 2 \end{bmatrix}$$

(good way to make sure you're right is @ this step... know you should get all 0's & same # on diagonal.)

$$= \begin{bmatrix} -13 & 0 & 0 \\ 0 & -13 & 0 \\ 0 & 0 & -13 \end{bmatrix} = -13 * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -13 * I$$

$\det(A) = -13.$

So, $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{-13} \begin{bmatrix} 7 & 3 & 5 \\ -4 & 2 & -1 \\ -5 & -4 & 2 \end{bmatrix}.$

4. Find the determinant of $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 5 & 3 \end{bmatrix}$.

* can choose any row or column to do cofactor expansion. Choose signs according to: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$. *

$$\det A = 3\begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 5 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}$$

$$= 3[3-10] - [6-20] + [4-4]$$

$$= 3(-7) + 14 = -21 + 14 = \boxed{-7}.$$

5. (a) Solve the DE $x^2y'' + 10xy' + 8y = x^2$.

Method 1: Sub. $y = x^m$:

$$x^m\left[m(m-1) + 10m + 8\right] = 0$$

$$m^2 + 9m + 8 = 0$$
$$(m+8)(m+1) = 0$$
$$m = -8,\ m = -1.$$

$$y_c = c_1 \underbrace{x^{-1}}_{y_1} + c_2 \underbrace{x^{-8}}_{y_2}.$$

Now can use variation of parameters to find y_p:

$y'' + 10x^{-1}y' + 8x^{-2} = \underbrace{1}_{f(x)}.$

$$W = \begin{vmatrix} x^{-1} & x^{-8} \\ -x^{-2} & -8x^{-9} \end{vmatrix} = -8x^{-10} + x^{-10} = -7x^{-10}.$$

$$W_1 = \begin{vmatrix} 0 & x^{-8} \\ 1 & -8x^{-9} \end{vmatrix} = -x^{-8}. \quad W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & 1 \end{vmatrix} = x^{-1}.$$

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{1}{7}x^2\,dx = \frac{1}{21}x^3.$$

$$u_2 = \int \frac{W_2}{W}\,dx = \frac{-1}{7}x^9 = \frac{-1}{70}x^{10}.$$

$$y_p = u_1y_1 + u_2y_2 = \frac{1}{21}x^2 - \frac{1}{70}x^2 = \frac{1}{30}x^2.$$

$$\boxed{\therefore\ y = c_1x^{-1} + c_2x^{-8} + \frac{1}{30}x^2.}$$ General solution

Method 2: Sub. $x = e^t$.

$a_2 = 1, \ a_1 = 10, \ a_0 = 8.$

$$a_2 \tilde{y}'' + (a_1 - a_2)\tilde{y}' + a_0 \tilde{y} = 0$$

$\tilde{y}'' + 9\tilde{y}' + 8\tilde{y} = 0$ linear homog. constant coef.

$m^2 + 9m + 8 = 0$

$(m+8)(m+1) = 0$

$m = -8, \ m = -1.$

$\tilde{y}_c = c_1 e^{-8t} + c_2 e^{-t}.$

$\Big\downarrow$ sub. back $t = \ln x$

$y_c = c_1 x^{-8} + c_2 x^{-1}.$

To solve for y_p, can use y_c & variation of parameters as on previous page, or use $\tilde{y}_c$ & plug $x = e^t$ into RHS:

$\tilde{y}'' + 9\tilde{y}' + 8\tilde{y} = e^{2t}. \qquad \tilde{y}_c = c_1 e^{-8t} + c_2 e^{-t}.$

Undetermined coef.: Guess: $\tilde{y}_p = Ae^{2t}$

$\tilde{y}_p' = 2Ae^{2t}$

$\tilde{y}_p'' = 4Ae^{2t}.$

$4Ae^{2t} + 18Ae^{2t} + 8Ae^{2t} = e^{2t} \Rightarrow 30A = 1 \Rightarrow A = \frac{1}{30}.$

So, $\tilde{y}_p = \frac{1}{30} e^{2t} \xRightarrow[t = \ln x]{} y_p = \frac{1}{30} x^2.$

$\therefore y = y_c + y_p = c_1 x^{-8} + c_2 x^{-1} + \frac{1}{30} x^2.$

(b) Solve $x^2y'' + xy' + 4y = 0.$

$x^m[m(m-1) + m + 4] = 0$

$\Rightarrow m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow$ $m = \pm 2i.$

i.e. $\alpha = 0,\ \beta = 2.$

$\therefore \boxed{y = c_1\cos(2\ln x) + c_2\sin(2\ln x)}.$

(c) Solve $x^3y''' - 6y = 0.$

$x^m[m(m-1)(m-2) - 6] = 0$

$\Rightarrow (m^2 - m)(m-2) - 6 = 0 \Rightarrow m^3 - 2m^2 - m^2 + 2m - 6 = 0$

$\Rightarrow m^3 - 3m^2 + 2m - 6 = 0.$ $m = 3$ works.

$$\begin{array}{r|l} & m^2 + 2 \\ \hline m-3 & m^3 - 3m^2 + 2m - 6 \\ & \underline{m^3 - 3m^2} \\ & \qquad\quad 2m - 6 \\ & \qquad\quad \underline{2m - 6} \\ & \qquad\qquad\quad 0 \end{array}$$

$(m-3)(m^2+2)$

$m = 3.$ $m = \pm\sqrt{2}i.$

$y = c_1x^3 + c_2\cos(\sqrt{2}\ln x) + c_3\sin(\sqrt{2}\ln x).$

(d) $x^2y'' - xy' + y = 0.$

$x^m[m(m-1) - m + 1] = 0$

$\Rightarrow m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$ double root

$\therefore y = c_1x + c_2x\ln x.$

6. When a mass of 2 Kg is attached to a
[38, #33, spring whose constant is 32 N/m, it comes
34]. to rest in the equilibrium position. Starting
at $t=0$, a force equal to $F(t) = 68e^{-2t}\cos(4t)$
is applied to the system.

(a) Find the eq'n of motion in the absence of damping.

(b) What is the amplitude of vibrations after a very long time?

Spring/Mass systems w/ driven motion are described by the DE:

$$\underbrace{m}_{\text{Mass}} x'' = -\underbrace{Kx}_{\text{spring constant}} - \overbrace{\beta}^{\text{damping}} x' + \underbrace{F(t)}_{\text{external force}}. \Leftrightarrow x'' + 2\lambda x' + \underbrace{\omega^2}_{\sqrt{K/m}} x = F(t).$$

Here $m = 2$, $K = 32$, $\beta = 0$, $F(t) = 68e^{-2t}\cos(4t)$,
$x(0) = 0$, $x'(0) = 0$.

Solution to corr. homog. eq'n is: $x_c(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$,
since $\beta = 0$. Here $\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{32}{2}} = \sqrt{16} = 4$.

$$\therefore\ x_c(t) = c_1\cos(4t) + c_2\sin(4t).$$

$$x'' + 16x = 34e^{-2t}\cos(4t).$$

To find $x_p(t)$ use undetermined coef.:

$34e^{-2t} \Leftrightarrow Ae^{-2t}$

$\cos(4t) \Leftrightarrow B\cos(4t) + C\sin(4t)$.

So, $34e^{-2t}\cos(4t) \Leftrightarrow Be^{-2t}\cos(4t) + Ce^{-2t}\sin(4t) = x_p$.

$$x_p' = -2e^{-2t}\left[B\cos(4t) + C\sin(4t)\right] + e^{-2t}\left[-4B\sin(4t) + 4C\cos(4t)\right]$$

$$= e^{-2t}\left[(-2B + 4C)\cos(4t) + (-2C - 4B)\sin(4t)\right].$$

$$x_p'' = -2e^{-2t}\left[\,''\,\right] + e^{-2t}\left[(8B - 16C)\sin(4t) + (-8C - 16B)\cos(4t)\right]$$

$$= e^{-2t}\left[(4B - 8C - 8C - 16B)\cos(4t) + (4C + 8B + 8B - 16C)\sin(4t)\right]$$

$$= e^{-2t}\left[(-12B - 16C)\cos(4t) + (16B - 12C)\sin(4t)\right].$$

$$x_p'' + 16x_p = 34e^{-2t}\cos(4t)$$

$\Rightarrow$ $-12B - 16C + 16B = 34$ & $16B - 12C + 16C = 0$

$4B - 16C = 34$ $\qquad$ $16B + 4C = 0$

$-17C = 34$ $\qquad$ $C = -4B$

$C = -2.$ $\qquad$ $-2 = -4B$

$B = \frac{1}{2}.$

$$\therefore x_p = e^{-2t}\left[\tfrac{1}{2}\cos(4t) - 2\sin(4t)\right].$$

$\therefore$ Eq^n of motion is: $x(t) = x_c + x_p = c_1\cos(4t) + c_2\sin(4t) + x_p.$

$x(0) = 0 \Rightarrow 0 = c_1 + \left[\frac{1}{2}\right] \Rightarrow c_1 = -\frac{1}{2}.$

$x'(0) = 0 \Rightarrow 4c_2 - 2\left[\frac{1}{2}\right] + (-8) = 0 \Rightarrow c_2 = \frac{9}{4}.$

(a) $$\boxed{\therefore x(t) = -\tfrac{1}{2}\cos(4t) + \tfrac{9}{4}\sin(4t) + e^{-2t}\left[\tfrac{1}{2}\cos(4t) - 2\sin(4t)\right].}$$

(b) As $t \to \infty$, $x_p \to 0$, since $e^{-2t} \to 0$. $\therefore$ we look at the terms $-\frac{1}{2}\cos(4t) + \frac{9}{4}\sin(4t) = A\sin(\omega t + \phi)$ where $A = \sqrt{c_1^2 + c_2^2}$ is the amplitude. $\therefore A = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{9}{4}\right)^2}$ $= \sqrt{\frac{1}{4} + \frac{81}{16}} = \sqrt{\frac{85}{16}} = \boxed{\frac{\sqrt{85}}{4}}.$