Math 2Z03 - Tutorial #6

1. For each of the following, the general solution of the associated homog. DE is given. What form will its particular solution have?

(a) $y'' + 3y = -48x^2 e^{3x}$; $y_c = c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)$.

$-48x^2 \longleftrightarrow a_2x^2 + a_1x + a_0$
$e^{3x} \longleftrightarrow Ae^{3x}$.

Trial solution: $(a_2x^2 + a_1x + a_0)Ae^{3x} = \underbrace{(Bx^2 + Cx + D)e^{3x}}_{\text{doesn't contain any terms } y_c \text{ does.}}$.

$\therefore$ The particular solution has the form $(Bx^2 + Cx + D)e^{3x}$, for some constants B, C, D.

(b) $y''' - 6y'' = 3 - \cos x$; $y_c = c_1 + c_2x + c_3e^{6x}$.

$3 \leftrightarrow a_0$
$-\cos x \leftrightarrow A\cos x + B\sin x$.

Trial solution: $\underbrace{a_0}_{}$ $+ A\cos x + B\sin x$.

$\updownarrow$ duplicates in y_c [corr. to c_1]. So multiply a_0 by x^2 [if multiply by x, $a_0x \leftrightarrow c_2x$].

$\therefore$ Particular solution has form $a_0x^2 + A\cos x + B\sin x$.

(c) $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$; $y_c = c_1e^{x/2} + c_2xe^{x/2}$.

$3 \leftrightarrow a_0$
$e^{x/2} \leftrightarrow Ae^{x/2}$. [Need to multiply by x^2, b/c of $c_1e^{x/2} + c_2xe^{x/2}$].

$\therefore$ Particular solution has form $a_0 + Ax^2e^{x/2}$.

2. Find the general solution of [a] - [c] by solving for the constants in the particular solution.

$\boxed{a}$ $y'' + 3y = -48x^2e^{3x}$.

$y_p = (Bx^2 + Cx + D)e^{3x}$.
$y_p' = (2Bx + C)e^{3x} + 3(Bx^2 + Cx + D)e^{3x}$
$y_p'' = (2B)e^{3x} + 3(2Bx + C)e^{3x} + 3(2Bx + C)e^{3x} + 9(Bx^2 + Cx + D)e^{3x}$.

$y_p'' + 3y_p = -48x^2e^{3x}$

$\Rightarrow e^{3x}[2B + 6Bx + 3C + 6Bx + 3C + 9Bx^2 + 9Cx + 9D + 3Bx^2 + 3Cx + 3D] = -48x^2e^{3x}$

$\Rightarrow 2B + 6C + 12D = 0$ & $12B + 12C = 0$ & $12B = -48$

$D = -\frac{1}{6}B - \frac{1}{2}C$ $\quad C = -B$ $\quad B = -4$
$D = \frac{4}{6} - 2 = \frac{2}{3} - \frac{6}{3} = -\frac{4}{3}$. $\quad C = 4$

$\therefore y_p = [-4x^2 + 4x - 4/3]e^{3x}$.

$\therefore y = c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x) + [-4x^2 + 4x - \frac{4}{3}]e^{3x}$.

$\boxed{b}$ $y''' - 6y'' = 3 - \cos x$.

$y_p = a_0x^2 + A\cos x + B\sin x$
$y_p' = 2a_0x - A\sin x + B\cos x$
$y_p'' = 2a_0 - A\cos x - B\sin x$
$y_p''' = A\sin x - B\cos x$.

$y_p''' - 6y_p'' = 3 - \cos x \Leftrightarrow A\sin x - B\cos x - 12a_0 + 6A\cos x + 6B\sin x = 3 - \cos x$

$\Rightarrow -12a_0 = 3$ & $A + 6B = 0$ & $-B + 6A = -1$
$a_0 = -\frac{1}{4}$ $\quad A = -6B$ $\quad -B - 36B = -1 \Rightarrow B = \frac{1}{37}$.
$\therefore y = c_1 + c_2x + c_3e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$.

$\boxed{c}$ $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$; $y_c = c_1 e^{x/2} + c_2 x e^{x/2}$.

$y_p = a_0 + Ax^2 e^{x/2}$.

$y_p' = 2Axe^{x/2} + \frac{1}{2}Ax^2 e^{x/2}$.

$y_p'' = 2Ae^{x/2} + Axe^{x/2} + Axe^{x/2} + \frac{1}{4}Ax^2 e^{x/2} = 2Ae^{x/2} + 2Axe^{x/2} + \frac{1}{4}Ax^2 e^{x/2}$.

$y_p'' - y_p' - \frac{1}{4}y_p = 3 + e^{x/2} \iff e^{x/2}\left[\frac{1}{4}Ax^2 + 2Ax + 2A - 2Ax - \frac{1}{2}Ax^2 + \frac{1}{4}Ax^2\right] + \frac{1}{4}a_0 = 3 + e^{x/2}$

$\iff$ $\underbrace{\frac{1}{4}a_0 = 3}_{a_0 = 12}$ $\quad$ $\underbrace{2A = 1}_{A = \frac{1}{2}}$

$\therefore\ y = c_1 e^{x/2} + c_2 x e^{x/2} + 12 + \frac{1}{2}x^2 e^{x/2}$.

3. Find the general solution of $x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2}$, where $y_1 = x^{-1/2}\cos x$ & $y_2 = x^{-1/2}\sin x$ are linearly independent solutions of the assoc. homog. DE on $(0,\infty)$.

$$y_c = c_1 \underbrace{x^{-1/2}\cos x}_{y_1} + c_2 \underbrace{x^{-1/2}\sin x}_{y_2}.$$

$$W = \begin{vmatrix} x^{-1/2}\cos x & x^{-1/2}\sin x \\ -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x & -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x \end{vmatrix}$$

$$= x^{-1}!$$

$$y'' + \frac{1}{x}y' + \frac{(x^2 - \frac{1}{4})}{x^2}y = \underbrace{x^{-1/2}}_{f(x)}.$$

$$W_1 = \begin{vmatrix} 0 & x^{-1/2}\sin x \\ x^{-1/2} & \end{vmatrix} = -x^{-1}\sin x.$$

$$W_2 = \begin{vmatrix} x^{-1/2}\cos x & 0 \\ & x^{-1/2} \end{vmatrix} = x^{-1}\cos x.$$

$$u_1 = \int -\sin x \, dx = \cos x.$$

$$u_2 = \int \cos x \, dx = \sin x.$$

$$y = c_1 x^{-1/2}\cos x + c_2 x^{-1/2}\sin x + x^{-1/2}.$$

4. Solve $y''' + y' = \tan x$ by variation of parameters.

$m^3 + m = 0 \iff m(m^2+1) = 0 \iff m=0$ or $m = \pm i$.

$y_c = \underbrace{c_1}_{y_1} + c_2 \underbrace{\cos x}_{y_2} + c_3 \underbrace{\sin x}_{y_3}$.

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1.$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x.$$

$W_2 = -\tan x \cos x = -\sin x$.

$W_3 = \tan x(-\sin x)$.

$u_1 = \int \tan x \, dx = -\ln|\cos x|$.

$u_2 = \int -\sin x \, dx = \cos x$.

$$u_3 = \int -\frac{\sin^2 x}{\cos x}\, dx = \int \frac{-1+\cos^2 x}{\cos x}\, dx = \int -\sec x + \cos x \, dx$$

$$= -\ln|\sec x + \tan x| + \sin x.$$

$y = c_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| + 1 - \sin x \ln|\sec x + \tan x|$

$= c_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| - \sin x \ln|\sec x + \tan x|.$