Math 2Z03 - Tutorial #5

1. Find the largest interval on which the IVP is guaranteed to have a unique solution, by Theorem 3.1.1.

$\boxed{a}$ $(x^2-1)y'' + 3xy' + \cos x\, y = e^x$, $y(0)=4$, $y'(0)=5$.

Recall: Theorem 3.1.1 [Linear Existence/Uniqueness]:

Consider the nth-order linear IVP $a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = g(x)$, $y(x_0)=y_0$, $y'(x_0)=y_1, \dots, y^{(n-1)}(x_0)=y_{n-1}$.

Suppose $a_n(x), \dots, a_0(x), g(x)$ are continuous on an interval I & $a_n(x) \neq 0$ $\forall x \in I$. If $x_0 \in I$, then this IVP has a unique solution on I.

Here x^2-1, $3x$, $\cos x$, & e^x are cont. everywhere.
$a_n(x) = a_2(x) = x^2-1 = 0 \iff x = \pm 1$. Here $x_0 = 0$.

So, need an I that contains $x_0 = 0$ & does not contain ± 1. $\therefore$ $I = (-1, 1)$ is the largest interval where we're guaranteed a unique solution by Theorem 3.1.1.

$\boxed{b}$ $\ln(x)y' + y = \cot(x)$, $y(2) = 3$.

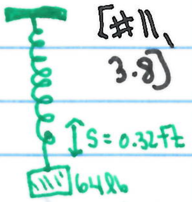
$\ln(x) = 0$ when $x = 1$, so I can't contain 1.
$x_0 = 2$, so I must contain 2.

$\ln(x)$ cont. for $x > 0$.
1 cont. everywhere.
$\cot(x) = \frac{\cos x}{\sin x}$ cont. when $\sin x \neq 0 \iff x \neq \pi n$, $n \in \mathbb{Z}$.

So, the largest interval containing 2 & not containing 1 & πn for $n \in \mathbb{Z}$ & not containing any $x \leq 0$ is $(1, \pi)$.

$\therefore$ The largest interval for which Theorem 3.1.1 guarantees a unique solution is $(1, \pi)$.

2. A mass weighing 64 lb stretches a spring 0.32 ft. [#11, 3.8] The mass is initially released from a pt 8 inches above the equilibrium position with downward velocity of 5 ft/s.



$x(0) = -\frac{8}{12}$ ft $= -\frac{2}{3}$ ft.

$x'(0) = 5$ ft/s.

(a) Find the equation of motion. $64 = mg \Rightarrow m = \frac{64}{\underbrace{32}_{ft/s^2}} = 2$.

By Hooke's Law, $F = ks \Rightarrow 64 = k\left(\frac{32}{100}\right) \Rightarrow 64 \cdot \frac{25}{8} = k \Rightarrow k = 200$ lb/ft.

at equilibrium $\underbrace{mg}_{weight} - ks = 0$

$\frac{4}{25} \cdot 8 = 200$

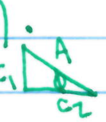
So, $m\frac{d^2x}{dt^2} = \underbrace{-kx}_{-k(s+x)+mg} \Rightarrow \frac{d^2x}{dt^2} + \frac{200}{2}x = 0 \Rightarrow \frac{d^2x}{dt^2} + \underbrace{100}_{w^2}x = 0 \Rightarrow w = 10$.

$x(t) = c_1\cos(wt) + c_2\sin(wt) = c_1\cos(10t) + c_2\sin(10t)$.

$x(0) = -\frac{2}{3} \Rightarrow c_1 = -\frac{2}{3}$. $x'(0) = 5 \Rightarrow 5 = 10c_2 \Rightarrow c_2 = \frac{1}{2}$.

$\therefore \boxed{x(t) = -\frac{2}{3}\cos(10t) + \frac{1}{2}\sin(10t)}$

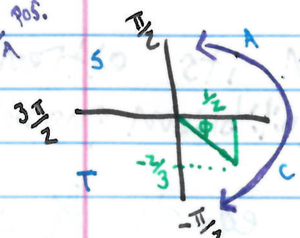
Let's also write it in the form $x = A\sin(wt + \phi)$.

Recall: Setting $c_1 = A\sin\phi$, $c_2 = A\cos\phi$, we have $c_1\cos(wt) + c_2\sin(wt)$
$= A\sin\phi\cos(wt) + A\cos\phi\sin(wt) = A\sin(wt+\phi)$.
Setting $\sin\phi = \frac{c_1}{A}$ & $\cos\phi = \frac{c_2}{A}$ gives a triangle .
So, $A = \sqrt{c_1^2 + c_2^2}$.

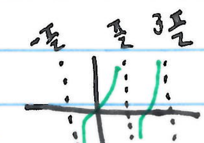
$\sin\phi = \frac{-2/3}{A}$ neg.

$\cos\phi = \frac{1/2}{A}$ pos.

Here $A = \sqrt{\frac{4}{9} + \frac{1}{4}} = \sqrt{\frac{25}{36}} = \frac{5}{6}$. $c_1 = -\frac{2}{3}$. $c_2 = \frac{1}{2}$.



Putting $\arctan\left(\frac{c_1}{c_2}\right) = \arctan\left(-\frac{4}{3}\right)$ into calculator spits out a ϕ b/w $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by default. In this example, our ϕ lives in the 4th quadrant, so this is okay. [If ϕ was in the 2nd or 3rd quadrant, we would have to add π to $\arctan\left(\frac{c_1}{c_2}\right)$.



$\arctan\left(-\frac{4}{3}\right) = -0.9273$. $\therefore \boxed{x(t) = \frac{5}{6}\sin(10t - 0.927)}$.

b What is the amplitude & period of motion?

Given $x(t) = A\sin(\omega t + \phi)$, A is the amplitude & the period is $\frac{2\pi}{\omega}$.

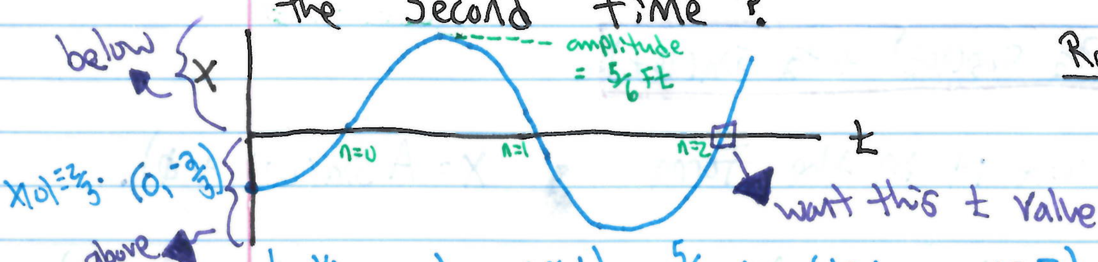
So, here the amplitude is $\frac{5}{6}$ ft & the period is $\frac{2\pi}{10} = \frac{\pi}{5}$.

c How many complete cycles will the mass have completed at the end of 3π seconds?

The period is $\pi/5$. This means completes 1 cycle in $\pi/5$ seconds.

$\frac{3\pi}{\pi/5} = 15$. $\therefore$ 15 cycles in 3π seconds.

d At what time does the mass pass through the equilibrium position heading downward for the second time?



Recall: below equilibrium is positive direction & above it is negative.

Looking at $x(t) = \frac{5}{6}\sin(10t - 0.927)$, we'll have $x(t) = 0$ when $10t - 0.927$ is an integer multiple of π. Let $n \in \mathbb{Z}$.

$$10t - 0.927 = n\pi \Rightarrow t = \frac{n\pi}{10} + \frac{0.927}{10}.$$

$n=2$: $\frac{2\pi}{10} + \frac{0.927}{10} = \boxed{0.721 \text{ seconds}}$.

e At what time does the mass attain its extreme displacement on either side of the equilibrium position?

Want to know for what t values $x(t)$ equals $\frac{5}{6}$ & $-\frac{5}{6}$.

$x(t) = \frac{5}{6}\sin(10t - 0.927) = \pm\frac{5}{6}$ when $10t - 0.927 = \frac{\pi}{2} + n\pi$ for $n \in \mathbb{Z}$. $\Leftrightarrow t = \frac{\pi}{20} + \frac{n\pi}{10} + \frac{0.927}{10}$, $n = 0, 1, 2, \ldots$

$\boxed{f}$ What is the position of the mass at $t = 3s$?

$X(3) = \frac{5}{6} \sin(30 - 0.927) = -0.597 \text{ Ft.}$

$\boxed{g}$ What is the instantaneous velocity at $t = 3s$?

$x'(t) = \frac{50}{6} \cos(10t - 0.927).$

$x'(3) = \frac{50}{6} \cos(30 - 0.927) = -5.814 \text{ ft/s}.$

$\boxed{h}$ What is the acceleration at $t = 3s$?

$x''(t) = -\frac{500}{6} \sin(10t - 0.927).$

$x''(3) = -\frac{500}{6} \sin(30 - 0.927) = 59.702 \text{ ft/s}^2.$

$\boxed{i}$ What is the instantaneous velocity at the times when the mass passes through the equilibrium position?

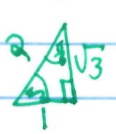
In $\boxed{d}$ we found that the mass passes through the equilibrium position for $t = \frac{n\pi}{10} + \frac{0.927}{10}, n = 0, 1, 2, \ldots$

$x'(t) = \frac{50}{6} \cos(10t - 0.927).$

$x'\left(\frac{n\pi}{10} + \frac{0.927}{10}\right) = \frac{50}{6} \cos(n\pi + 0.927 - 0.927) = \frac{50}{6} \cos(n\pi)$

$= \pm \frac{50}{6} \text{ ft/s} = \pm \frac{25}{3} \text{ ft/s}.$

$\boxed{j}$ At what times is the mass 5 inches below the equilibrium position?



$\frac{5}{12} = x(t) = \frac{5}{6} \sin(10t - 0.927) \Rightarrow \frac{1}{2} = \sin(10t - 0.927)$

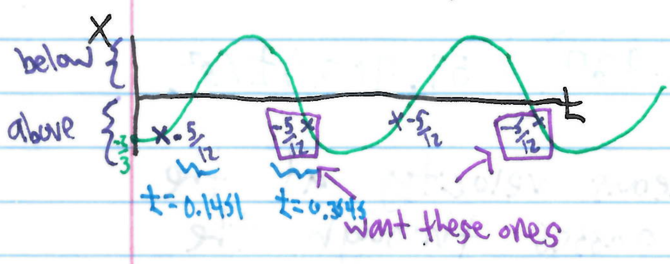
$\sin\theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6} + 2\pi n$ for $n \in \mathbb{Z}$
& when $\theta = \frac{5\pi}{6} + 2\pi n$ for $n \in \mathbb{Z}$.

So, need $10t - 0.927 = \frac{\pi}{6} + 2\pi n$, $n = 0, 1, 2, \dots$
or $10t - 0.927 = \frac{5\pi}{6} + 2\pi n$, $n = 0, 1, 2, \dots$.

$\Leftrightarrow t = \frac{\pi}{60} + \frac{\pi n}{5} + \frac{0.927}{10}$, $n = 0, 1, 2, \dots = 0.1451 + \frac{\pi n}{5}$, $n = 0, 1, 2, \dots$

or $t = \frac{\pi}{12} + \frac{\pi n}{5} + \frac{0.927}{10}$, $n = 0, 1, 2, \dots = 0.3545 + \frac{\pi n}{5}$, $n = 0, 1, 2, \dots$

$\boxed{K}$ At what times is the mass 5 inches below the equilibrium position heading in the upward direction?



$t = 0.3545 + \frac{\pi n}{5}$, $n = 0, 1, 2, \dots$.

3. A mass weighing 32 lb is suspended from a spring whose spring constant is 9 lb/ft. The mass is initially released from a pt 1 ft above the equilibrium position with an upward velocity of $\sqrt{3}$ ft/s. Find the times for which the mass is heading downward at a velocity of 3 ft/s.

[#12.3.8]

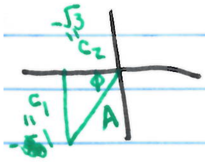
$x(0) = -1$.
$x'(0) = -\sqrt{3}$.

$m = 1$. $[32 = m(32)]$. $\frac{d^2x}{dt^2} = -9x$. $\omega = 3$.

$x(t) = c_1\cos(3t) + c_2\sin(3t)$.
$x(0) = -1 \Rightarrow c_1 = -1$. $x'(0) = -\sqrt{3} \Rightarrow -\sqrt{3} = 3c_2 \Rightarrow c_2 = \frac{-\sqrt{3}}{3}$.

$x(t) = -\cos(3t) - \frac{\sqrt{3}}{3}\sin(3t)$.

$A = \sqrt{c_1^2 + c_2^2} = \sqrt{1 + \frac{3}{9}} = \sqrt{\frac{12}{9}} = \frac{2}{3}\sqrt{3} = \frac{2}{\sqrt{3}}$.

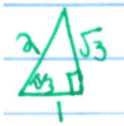


$$c_1 = A\sin\phi \Rightarrow \sin\phi = \frac{-1}{A} = -\frac{\sqrt{3}}{2} \quad \text{negative.}$$

$$c_2 = A\cos\phi \Rightarrow \cos\phi = -\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2} = -\frac{1}{2} \quad \text{negative.}$$

S	A
$\boxed{T}$	C

ϕ is in 4th quadrant.



$$\phi = \arctan\left(\frac{c_1}{c_2}\right) + \pi = \arctan(\sqrt{3}) + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}.$$

$$x(t) = \frac{2}{\sqrt{3}}\sin\left(3t + \frac{4\pi}{3}\right).$$

$$x'(t) = \frac{6}{\sqrt{3}}\cos\left(3t + \frac{4\pi}{3}\right).$$

$$3 = \frac{6}{\sqrt{3}}\cos\left(3t + \frac{4\pi}{3}\right) \Rightarrow \frac{\sqrt{3}}{2} = \cos\left(3t + \frac{4\pi}{3}\right).$$

$$\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6} + 2\pi n,\ n = 0,1,2,\ldots$$

$$\underline{\text{or}}\quad \theta = \frac{11\pi}{6} + 2\pi n,\ n = 0,1,2,\ldots.$$

$$3t + \frac{4\pi}{3} = \frac{\pi}{6} + 2\pi n \Rightarrow t = \frac{\pi}{18} + \frac{2\pi n}{3} - \frac{4\pi}{9} = \boxed{-\frac{7\pi}{18} + \frac{2\pi n}{3},\ n = 1,2,\ldots}$$

$$3t + \frac{4\pi}{3} = \frac{11\pi}{6} + 2\pi n \Rightarrow t = \frac{11\pi}{18} + \frac{2\pi n}{3} - \frac{4\pi}{9} = \boxed{\frac{\pi}{6} + \frac{2\pi n}{3},\ n = 0,1,2,\ldots}$$