Math 2Z03 - Tutorial #4

1. Is the set of functions $\{1+x, x, x^2\}$ linearly independent on $(-\infty, \infty)$?

Suppose $c_1(1+x) + c_2(x) + c_3(x^2) = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow c_3 x^2 + (c_1+c_2)x + c_1 = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow c_3 = 0$ & $c_1 + c_2 = 0$ & $c_1 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

$\Rightarrow$ linearly independent.

2. Suppose $f_1, f_2,$ & f_3 are solutions to a 2^{nd}-order linear homogeneous DE. Is $\{f_1, f_2, f_3\}$ a fundamental set of solutions?

The solution space has dimension 2, since the DE is 2^{nd}-order. $\therefore$ Any basis has exactly 2 elements $\Rightarrow \{f_1, f_2, f_3\}$ is a linearly dependent set & therefore is <u>not</u> a fundamental set of solutions.

3. The functions e^t & te^t satisfy the DE $y'' - 2y' + y = 0$. Is $y = c_1 e^t + c_2 t e^t$ a general solution of this DE?

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

Since we know e^t & te^t are solutions, it suffices to check to see that the Wronskian is nonzero.

$$W(e^t, te^t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t}$$

$$= e^{2t} \neq 0 \quad \forall t \in \mathbb{R}$$

$\Rightarrow te^t$ & e^t linearly independent $\Rightarrow y = c_1 e^t + c_2 t e^t$ general solution.

4. Find the general solution of $\underbrace{y^{(3)} + 8y = 0}_{\text{homogeneous linear w/ constant coef.}}$.

Recall: We plug $y = e^{mx}$ into the eq'n & Find the roots of the corresponding aux. eq'n.

$e^{mx}[m^3 + 8] = 0 \Rightarrow m^3 + 8 = 0 \Rightarrow m^3 = -8.$

* Can go about Finding the roots of this in a variety of ways. *

Method 1: Find one root α, then divide m^3+8 by $(m-\alpha)$.

By inspection, we can see that $m=-2$ is a root. A more systematic approach is given by the rational roots test:

Rational Roots Test [pg. 122]: If $m_1 = p/q$ is a rational root (expressed in lowest terms) of an aux. eq'n with integer coef. $a_n m^n + \dots + a_1 m + a_0 = 0$, then p is a factor of a_0 & q is a factor of a_n.

Here $a_0 = 8 \Rightarrow p$ must be either ± 1 or ± 2 or ± 4 or ± 8.
Here $a_n = a_3 = 1 \Rightarrow q = \pm 1$. We test these p's & see -2 works.

$$\begin{array}{r} m^2-2m+4 \\ m+2\overline{\big)\,m^3+8} \\ \underline{m^3+2m^2} \\ -2m^2+8 \\ \underline{-2m^2-4m} \\ 4m+8 \\ \underline{4m+8} \\ 0 \end{array}$$

So, $m^3+8 = (m+2)(m^2-2m+4)$.

$$m = \frac{2 \pm \sqrt{4-4(4)}}{2} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}\,i.$$

$\therefore$ Roots are $m=-2$, $m = 1+\sqrt{3}\,i$ & $m = 1-\sqrt{3}\,i$.
[Poly. has deg. 3, so know have exactly 3 roots, & complex roots come in conj. pairs].

Recall: $\boxed{i}$ If there are j distinct roots $m_1, \dots, m_j$, then the general solution contains the linear combo. $c_1 e^{m_1 x} + \dots + c_j e^{m_j x}$.

$\boxed{ii}$ If m_1 is a root of multiplicity q then the general solution contains the linear combo. $K_1 e^{m_1 x} + K_2 x e^{m_1 x} + \dots + K_q x^{q-1} e^{m_1 x}$.

$\boxed{iii}$ Using Euler's formula, we can write $c_1 e^{m_1 x} + c_2 e^{m_2 x}$ as $K_1 e^{\alpha x}\cos(\beta x) + K_2 e^{\alpha x}\sin(\beta x)$, where $m_1 = \alpha + \beta i$ & $m_2 = \alpha - \beta i$.

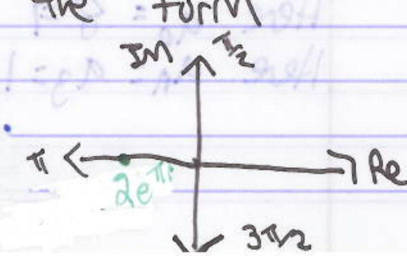
Here we have one distinct root $m=-2$ & a complex conj. pair $m = 1 \pm \sqrt{3}\,i$, so our general solution is

$$y = c_1 e^{-2x} + e^{x}\left[c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)\right].$$

Method 2: We know there will be 3 roots [possibly with mult. >1] of the form $re^{i\theta} = r(\cos\theta + i\sin\theta)$.

$$m^3 = -8.$$

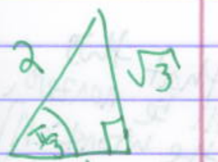
As a complex number, $-2 = 2e^{\pi i}$.



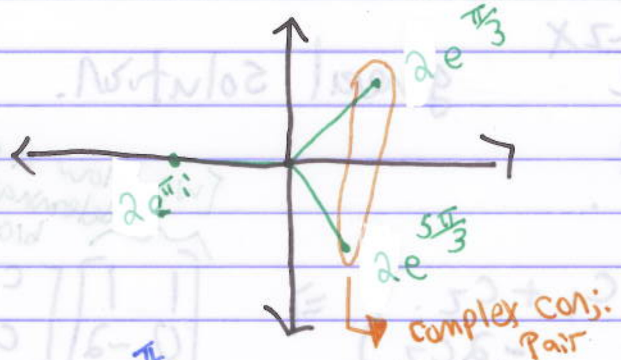
The other 2 roots will be equally spaced in the complex plane $\Rightarrow \frac{2\pi}{3}$ apart.

$\pi + \frac{2\pi}{3} = \frac{5\pi}{3}.$

$\pi - \frac{2\pi}{3} = \frac{\pi}{3}.$



$\cos(\pi/3) = 1/2$
$\sin(\pi/3) = \sqrt{3}/2.$



$\therefore$ Our 3 roots are $2e^{\pi i}$, $2e^{\pi/3}$, & $2e^{5\pi/3}$.

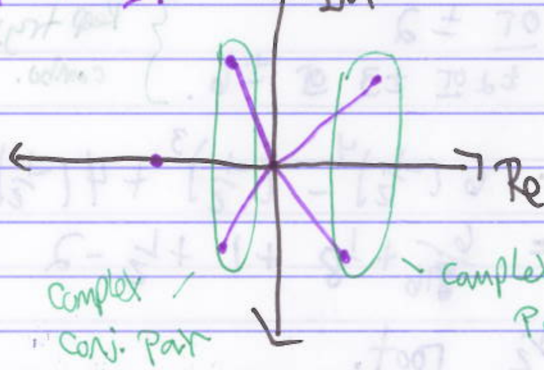
$2e^{\pi/3} = 2[\cos(\pi/3) + i\sin(\pi/3)] = 2[\frac{1}{2} + \frac{\sqrt{3}}{2} i] = 1+\sqrt{3}i$
$2e^{5\pi/3} = 2[\cos(5\pi/3) + i\sin(5\pi/3)] = 2[\frac{1}{2} - \frac{\sqrt{3}}{2} i] = 1-\sqrt{3}i.$

$\therefore$ Roots are -2, $1+\sqrt{3}i$, & $1-\sqrt{3}i$.

More systematically: we suppose $m = re^{i\theta}$. We know $-8 = 8e^{\pi i}$. $m^3 = -8 \Rightarrow r^3 e^{3i\theta} = 8e^{\pi i}$
$\Rightarrow r^3 = 8$ & $3\theta = \pi + 2\pi k$ for $k = 0, 1, 2 \Rightarrow \theta = \pi/3, \pi + \frac{2\pi}{3}$, & $\pi + 4\pi/3$
$\Rightarrow m = 2e^{\frac{\pi}{3}i}, 2e^{\pi i}$, & $2e^{\frac{5\pi}{3}i}$.

<u>Exercise</u>: Find the 5 roots of $m^5 + 32 = 0$, & use this to solve $y^{(5)} + 32y = 0$.

[Using the method above you'll find $2e^{\frac{\pi}{5}i}$, $2e^{\frac{3\pi}{5}i}$, $2e^{\pi i}$, $2e^{\frac{7\pi}{5}i}$, $2^{\frac{9\pi}{5}i}$].



5. Solve the IVP $y'' + 2y' = 0$, $y(0) = 1$, $y'(0) = 1$.

$m^2 + 2m = 0 \Rightarrow m(m+2) = 0 \Rightarrow m = 0$ & $m = -2$.

$y = c_1 + c_2 e^{-2x}$ general solution.

$y' = -2c_2 e^{-2x}$.

$\begin{rcases} y(0)=1 \Rightarrow 1 = c_1 + c_2 \\ y'(0)=1 \Rightarrow 1 = -2c_2 \end{rcases} \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}}_{\text{[we know a priori that this determinant will be nonzero, b/c it's the Wronskian]}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$

$\Rightarrow c_1 = 3/2$ & $c_2 = -1/2$.

$\therefore y = 3/2 - \frac{1}{2}e^{-2x}$.

6. Find a general solution for $6y^{(4)} - y''' + 4y'' - y' - 2y = 0$.

$6m^4 - m^3 + 4m^2 - m - 2 = 0$.

Here $a_0 = -2$ & $a_n = a_4 = 6$. If there is a rational root $\frac{p}{q}$, then we know p is a factor of -2 & q is a factor of 6:

$\begin{rcases} p = \pm 1 \text{ or } \pm 2 \\ q = \pm 1 \text{ or } \pm 2 \text{ or } \pm 3 \text{ or } \pm 6. \end{rcases}$ Keep trying til you find a combo. that works

Try: $-\frac{1}{2}$: $6(-\frac{1}{2})^4 - (-\frac{1}{2})^3 + 4(-\frac{1}{2})^2 - (-\frac{1}{2}) - 2$

$= \frac{6}{16} + \frac{1}{8} + 1 + \frac{1}{2} - 2 = 0.$ ✓

So, $m = -1/2$ root.

$$\begin{array}{r} 6m^3 - 4m^2 + 6m - 4 \\ m+\tfrac{1}{2}\ \overline{\big)\,6m^4 - m^3 + 4m^2 - m - 2} \\ 6m^4 + 3m^3 \qquad\qquad\qquad \\ \hline -4m^3 + 4m^2 - m - 2 \\ -4m^3 - 2m^2 \qquad\quad \\ \hline 6m^2 - m - 2 \\ 6m^2 + 3m \quad\ \\ \hline -4m - 2 \\ -4m - 2 \\ \hline 0. \end{array}$$

So, $6m^4 - m^3 + 4m^2 - m - 2$

$= (m+\tfrac{1}{2})(6m^3 - 4m^2 + 6m - 4).$

$p = \pm 1, \pm 2, \pm 4$

$q = \pm 1, \pm 2, \pm 3, \pm 6.$

$2/3$ works.

$$\begin{array}{r} 6m^2 + 6 \qquad\qquad \\ m-\tfrac{2}{3}\ \overline{\big)\,6m^3 - 4m^2 + 6m - 4} \\ 6m^3 - 4m^2 \qquad\qquad \\ \hline 6m - 4 \\ 6m - 4 \\ \hline 0 \end{array}$$

$(m+\tfrac{1}{2})(m-\tfrac{2}{3})(6m^2+6).$

$6m^2 + 6 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i.$

$\therefore$ Four roots are $m = -\tfrac{1}{2}, \tfrac{2}{3}, i, \&\ -i.$

$$\therefore\quad y = c_1 e^{-x/2} + c_2 e^{2x/3} + c_3 \cos x + c_4 \sin x.$$