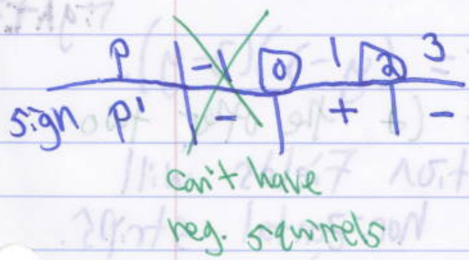
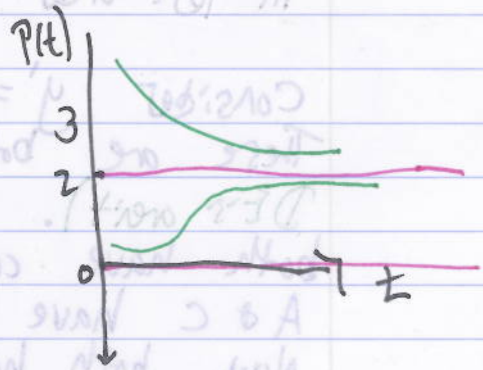
Math 2Z03 - Tutorial #3

1. Suppose that the population p (in thousands) of squirrels in Hamilton can be modelled by the DE $\frac{dp}{dt} = p(2-p)$.

(a) If the initial population of squirrels is 3000, what can you say about the long-term behaviour of the squirrel population?

$p(2-p) = 0 \iff p = 0$ or $p = 2$.





Squirrel population will tend to 2000 in the long-term.

(b) Can a population of 1000 ever decline to 500? Explain.

No, b/c solution curves strictly increasing b/w $0 < p < 2$, so if we begin w/ 1000 it'll never decrease to 500.

(c) Can a population of 1000 ever increase to 3000? Explain.

No, b/c a solution curve starting at $p=1$ can't cross the constant solution $p=2$, so it will tend to 2000, but can never reach 3000.

d How many squirrels will there be after one year if the initial population of squirrels is 50?

$P' = P(2-P)$.

$\int \frac{1}{P(2-P)}\,dP = \int dt$

$\frac{1}{P(2-P)} = \frac{A}{P} + \frac{B}{2-P} \Rightarrow 1 = A(2-P) + BP$

$\Rightarrow 1 = (-A+B)P + 2A \Rightarrow \underbrace{1 = 2A}_{A = 1/2}$ & $\underbrace{-A+B=0}_{B = A = 1/2}$.

$\int \frac{1}{2P} + \frac{1}{2(2-P)}\,dP = t + C$

$\frac{1}{2}\ln|P| - \frac{1}{2}\ln|2-P| = t + C$

$\Rightarrow \ln\left|\frac{P}{2-P}\right| = 2t + C \Rightarrow \frac{P}{2-P} = Ce^{2t}$

$\Rightarrow P = 2Ce^{2t} - PCe^{2t} \Rightarrow P = \frac{2Ce^{2t}}{1+Ce^{2t}}$.

50 squirrels $= \frac{1}{20}$ thousand squirrels.

So, $P(0) = \frac{1}{20}$.

$\frac{1}{20} = \frac{2C}{1+C} \Rightarrow \frac{1}{20}(1+C) = 2C \Rightarrow \frac{1}{20} = \frac{39}{20}C \Rightarrow C = \frac{1}{39}$.

So $P = \frac{2e^{2t}}{39(1+\frac{1}{39}e^{2t})} = \frac{2e^{2t}}{39+e^{2t}}$.

$$P(1) = \frac{2e^{2}}{39 + e^{2}}$$

So, after one year, there will be

$$\frac{2e^{2}}{39 + e^{2}} \approx 0.3186\ldots \text{ thousand squirrels}$$

$$\approx 319 \text{ squirrels.}$$

2. Consider the IVP $y' = 2x - 3y + 1$, $y(1) = 5$. Find an approximation of $y(1.2)$ using Euler's method with a step size of $h = 0.1$.

$$y_{n+1} = y_n + \frac{1}{10} F(x_n, y_n), \quad x_n = x_0 + nh.$$

Here $F(x,y) = 2x - 3y + 1$. $x_0 = 1$, $y_0 = 5$.

$\underline{n=1}$: $y_1 = y_0 + \frac{1}{10} F(x_0, y_0) = 5 + \frac{1}{10} F(1,5)$

$= 5 + \frac{1}{10}(2 - 15 + 1) = 5 + \frac{1}{10}(-12) = \frac{50}{10} - \frac{12}{10}$

$= \frac{38}{10} = \frac{19}{5}$. So, $x_1 = x_0 + h = 1 + \frac{1}{10} = \frac{11}{10}$.

$$\boxed{(x_1, y_1) = \left(\underbrace{\frac{11}{10}}_{1.1}, \underbrace{\frac{19}{5}}_{3.8}\right).}$$

$\underline{n=2}$: $x_2 = x_0 + 2\left(\frac{1}{10}\right) = \frac{10}{10} + \frac{2}{10} = \frac{12}{10} = 1.2$.

$y_2 = y_1 + \frac{1}{10} F(x_1, y_1) = \frac{19}{5} + \frac{1}{10}\left(\frac{22}{10} - \frac{57}{5} + 1\right)$

$= \frac{19}{5} + \frac{1}{10}\left(\frac{11}{5} - \frac{52}{5}\right) = \frac{19}{5} + \frac{1}{10}\left(-\frac{41}{5}\right)$

$= \frac{190}{50} - \frac{41}{50} = \frac{149}{50} = 2.98$. $(x_2, y_2) = (1.2, 2.98)$.

$\therefore y(1.2) \approx 2.98$.

3. $\boxed{a}$ Solve $xy' - y = 2x \ln x$.

$$y' \underbrace{- \tfrac{1}{x}}_{P(x)} y = \underbrace{2 \ln x}_{F(x)}$$

$e^{\int P(x)\,dx}$: $\int P(x)\,dx = \int -\frac{1}{x}\,dx = -\ln x$.

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} F(x)\,dx\right]$$

$$= e^{-(-\ln x)}\left[\int e^{-\ln x}(2 \ln x)\,dx\right]$$

$$= x\left[\int x^{-1}(2 \ln x)\,dx\right]$$

$$= 2x\left[\int \frac{\ln x}{x}\,dx\right] \qquad \begin{matrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{matrix}$$

$$= 2x\left[\int u\,du\right] = 2x\left[\tfrac{1}{2}u^2 + C\right]$$

$$= 2x\left[\tfrac{1}{2}(\ln x)^2 + C\right] = x(\ln x)^2 + Cx.$$

$$\therefore\ y = x(\ln x)^2 + Cx.$$

$\boxed{b}$ Find the largest interval where this solution is defined.

Recall: Need to find an interval where y is C^1 (continuously differentiable).

y defined for $x > 0$.

$y' = (\ln x)^2 + \frac{2x(\ln x)}{x} + C$ cont. on $(0, \infty)$.

$\therefore$ Solution defined on $(0, \infty)$.

4. A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing 1/2 pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

Let $A(t)$ denote the amount of salt in the tank at time t (measured in lb).

Want to find $A(30)$.

We know $A(0) = 10$ lb.

We also know:

$$\frac{dA}{dt} = \underbrace{\begin{pmatrix}\text{input rate}\\ \text{of salt}\end{pmatrix}}_{\substack{\| \\ R_{in}}} - \underbrace{\begin{pmatrix}\text{output rate}\\ \text{of salt}\end{pmatrix}}_{\substack{\| \\ R_{out}}}.$$

$$R_{in} = \underbrace{\begin{pmatrix}\text{concentration of}\\ \text{salt inflow}\\ \text{lb/gal}\end{pmatrix}}_{} * \underbrace{\begin{pmatrix}\text{input rate}\\ \text{of brine}\\ \text{gal/min}\end{pmatrix}}_{} = \begin{pmatrix}\text{input rate}\\ \text{lb/min}\end{pmatrix}.$$

Here: $\frac{1}{2}$ lb/gal $*$ 6 gal/min $=$ 3 lb/min.

So, $R_{in} = 3$ lb/min.

$$R_{out} = \underbrace{\begin{pmatrix}\text{Concentration of}\\ \text{salt outflow}\\ \text{lb/gal}\end{pmatrix}}_{\frac{A}{(\text{amount of brine})}} * \underbrace{\begin{pmatrix}\text{output rate}\\ \text{of brine}\\ \text{gal/min}\end{pmatrix}}_{4\text{ gal/min}} = \begin{pmatrix}\text{output rate}\\ \text{of salt}\\ \text{lb/min}\end{pmatrix}.$$

To find this, we need to know how much brine is in the tank at time t.

The liquid accumulates in the tank at the rate of $r_{in} - r_{out} = 6$ gal/min. $- 4$ gal/min $= 2$ gal/min.

$\therefore$ After t minutes there are $100 + 2t$ gallons of brine in the tank.

So, the concentration of outflow is $\frac{A}{100+2t}$.

$$\therefore R_{out} = \left(\frac{A}{100+2t}\text{ lb/gal}\right) * \left(4\text{ gal/min}\right) = \frac{4A}{100+2t}\text{ lb/min}.$$

$$\therefore \frac{dA}{dt} = 3 - \frac{4A}{100+2t}.$$

i.e., $A' + \underbrace{\frac{4}{100+2t}}_{P(x)} A = \underbrace{3}_{F(x)}.$ Linear

$$\int P(x)dx = \frac{4}{2}\ln|100+2t| = 2\ln|2t+100|.$$

$$A = e^{-\int P(x)}\left[\int e^{\int P(x)} F(x)\,dx\right]$$

$$= e^{\ln|(2t+100)^{-2}|}\left[3\int (2t+100)^2\,dx\right]$$

$u = 2t+100$
$du = 2dt$.

$$= 3(2t+100)^{-2}\left[\frac{1}{2\cdot 3}(2t+100)^3 + c\right]$$

$$= \frac{1}{2}(2t+100) + c(2t+100)^{-2}.$$

$A(0) = 10$

$\Rightarrow 10 = 50 + \frac{C}{100^2} \Rightarrow C = -40 \cdot 10000 = -400000.$

$\therefore A = t + 50 - 400{,}000\,(2t+100)^{-2}.$

$\therefore A(30) = 80 - 400{,}000(160)^{-2} = \frac{515}{8} = 64.375.$

After 30min. there will be ~64.38 lb of salt in the tank.