# Math 2203 - Tutorial #2

1. Solve the following DE's.

(a) $e^{-x^2} y' = e^{-y} x$.

$$\underbrace{e^{y}}_{1/h(y)} y' = \underbrace{xe^{x^2}}_{g(x)}$$

$$\int e^{y}\, dy = \int x e^{x^2}\, dx \qquad u = x^2$$

$$du = 2x\, dx$$

$$\Leftrightarrow e^{y} = \tfrac{1}{2}\int e^{u}\, du \qquad \underbrace{\tfrac{1}{2} du = x\, dx}_{\text{Substitution}}$$

$$\Leftrightarrow e^{y} = \tfrac{1}{2} e^{u} + c$$

$$\Leftrightarrow e^{y} = \tfrac{1}{2} e^{x^2} + c$$

$$\Leftrightarrow \boxed{y = \ln\left(\tfrac{1}{2} e^{x^2} + c\right)}.$$

**Should ask yourself:**

Is there a substitution I can make s.t. after I make the substitution, everything under the integral will be in terms of "u"?

(b) $\frac{1}{x} y' = y^{-1} e^{2x}$.

$$y y' = x e^{2x}$$

$$\int y\, dy = \int x e^{2x}\, dx$$

No substitution I can make to get everything in terms of u. e.g. I could try $u = 2x$, $\frac{1}{2} du = dx$, but then I would still have an "x" under $\int$.
∴ Use integration by parts.

$$\Leftrightarrow \tfrac{1}{2} y^2 = uv - \int v\, du$$

$$\Leftrightarrow \tfrac{1}{2} y^2 = \tfrac{1}{2} x e^{2x} - \tfrac{1}{2}\int e^{2x}\, dx$$

$$\Leftrightarrow y^2 = x e^{2x} - \left[\tfrac{1}{2} e^{2x} + c\right]$$

$$\Leftrightarrow \boxed{y^2 = x e^{2x} - \tfrac{1}{2} e^{2x} + c}.$$

Implicit solution

$$\underbrace{\begin{matrix} u = x & v = \tfrac{1}{2} e^{2x} \\ du = dx & dv = e^{2x}\, dx \end{matrix}}_{\text{Integration by Parts}}$$

L - Ln
I - Inverse
P - Polynomial
E - Exponential
T - Trig.

$\boxed{c}$ $y' = e^{x^2}$.

$\int dy = \underbrace{\int e^{x^2} dx}$

No way to integrate this by hand.

$\boxed{y = \int e^{x^2} dx}$. The anti-derivative is <u>not</u> an "elementary function".

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2. (a) Sketch the solution curves of $y' = \underbrace{y^2 - y^3}_{f(y)}$.

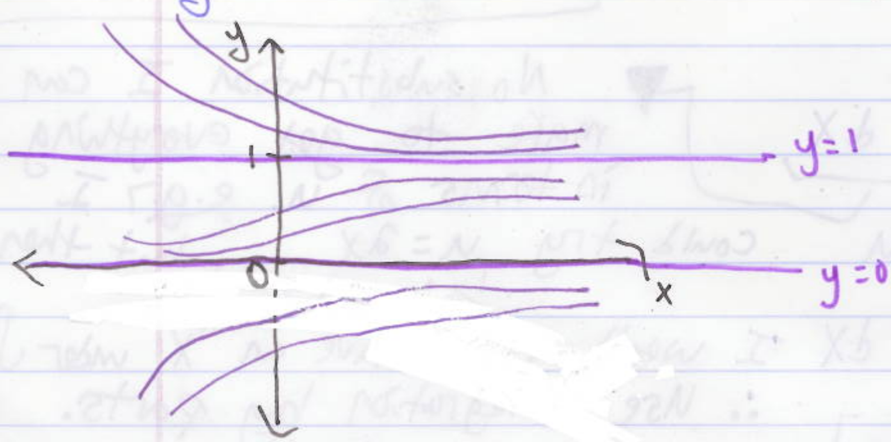
- <u>Critical points</u>: The zeros of $f(y)$:

$0 = y^2 - y^3 = y^2(1-y) \iff y = 0$ <u>or</u> $y = 1$.

- <u>Phase Portrait</u>:

| $y$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ | $2$ |
|---|---|---|---|---|---|
| sign of $y'$ | $+$ | | $+$ | | $-$ |


"1" stable (attractor).
"0" semi-stable.




$\boxed{b}$ Create an IVP involving this DE, s.t. in the long term the solution to the IVP approaches zero.

Can see from our sketch that for $y<0$ the

Solution curves approach zero as $x \to \infty$.
$\therefore\ y' = y^2 - y^2,\ y(0) = \underline{-1}$ would work.
[Could choose any negative value here].

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<u>Logistic Eq^n:</u> Models simple population growth.

- Suppose a population grows proportional to its size:

$$\frac{dP}{dt} = KP,\ K > 0.$$

<u>e.g.</u> - bacteria growth
- simple investment w/ compound interest

Solving this DE: $\int \frac{1}{P}\,dP = \int K\,dt \Leftrightarrow \ln P = Kt + C$

$\Leftrightarrow P = P_0 e^{Kt}$. * exponential growth *

- Similarly, $\frac{dP}{dt} = -KP,\ K > 0$ is * exponential decay *.

<u>e.g.</u> - radioactive decay
- breakdown of a chemical

In these 2 simple models, the growth (decay) is unbounded.

Suppose an environment can sustain no more than $\underbrace{\bar{P}}_{\text{carrying capacity}}$ individuals in its population.

When $P > \bar{P}$ we want $\frac{dP}{dt} < 0$. The DE that models this is: $\frac{dP}{dt} = KP(\bar{P} - P)$.

* make substitution $\bar{P} = \frac{a}{b},\ K = b$. *

$$\boxed{\frac{dP}{dt} = P(a-bP)}$$ Logistic Equation

Solving the Logistic Eq$^{n}$: It's autonomous & $\therefore$ separable.

[all autonomous DE's are separable].

$$\int dt = \underbrace{\int \frac{1}{P(a-bP)}\,dP}$$

[to integrate this we can use partial fractions]

$$\frac{1}{P(a-bP)} = \frac{A}{P} + \frac{B}{a-bP} = \frac{A(a-bP)+BP}{P(a-bP)}$$

$$\Leftrightarrow 1 = A(a-bP) + BP = P(-Ab+B) + aA$$

$$\Rightarrow \underbrace{aA=1}_{A=\frac{1}{a}} \quad \& \quad \underbrace{-Ab+B=0}_{B=\frac{b}{a}}.$$

So, $\dfrac{1}{P(a-bP)} = \dfrac{\frac{1}{a}}{P} + \dfrac{\frac{b}{a}}{a-bP}$.

$$\therefore \int dt = \int\left(\frac{\frac{1}{a}}{P} + \frac{\frac{b}{a}}{a-bP}\right)dP \qquad \boxed{\begin{aligned} u &= a-bP \\ du &= -b\,dP \\ -\tfrac{1}{b}\,du &= dP \end{aligned}}$$

$$\Leftrightarrow t = \tfrac{1}{a}\ln|P| + \tfrac{b}{a}\ln|a-bP|\cdot\left(\tfrac{-1}{b}\right) + C$$

$$\Leftrightarrow t + C = \tfrac{1}{a}\left[\ln|P| - \ln|a-bP|\right]$$

$$\Leftrightarrow a(t+c) = \ln\left|\frac{P}{a-bP}\right| \Leftrightarrow c_1e^{at} = \frac{P}{a-bP}$$

$$\Leftrightarrow ac_1e^{at} - bc_1e^{at}P = P \Leftrightarrow P = \frac{c_1ae^{at}}{1+bc_1e^{at}} \cdot \frac{\frac{1}{c_1e^{at}}}{\frac{1}{c_1e^{at}}}$$

$\Leftrightarrow P = \dfrac{a c_1}{e^{-at} + b c_1}$. Suppose $P(0) = P_0$, $P_0 \neq \frac{a}{b}$.

Subbing this in & solving for $c_1$:

$$P_0 = \frac{a c_1}{1 + b c_1} \Leftrightarrow P_0 + b c_1 P_0 = a c_1 \Leftrightarrow P_0 = (a - bP_0) c_1$$

$$\Leftrightarrow c_1 = \frac{P_0}{a - bP_0}.$$

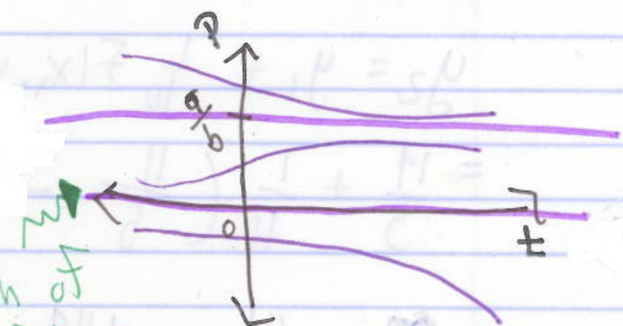
Subbing this back into the eq$^{n}$:

$$P = \frac{\frac{aP_0}{a-bP_0}}{e^{-at} + \frac{bP_0}{a-bP_0}} = \frac{aP_0}{a-bP_0} \cdot \frac{a-bP_0}{(a-bP_0)e^{-at} + bP_0}.$$

$$\boxed{\therefore P = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}}}.$$

From the solution, can see that as $t \to \infty$ $P \to \frac{aP_0}{bP_0} = \frac{a}{b}$.

As $t \to -\infty$, $P \to 0$.




Sketch of solution curves.

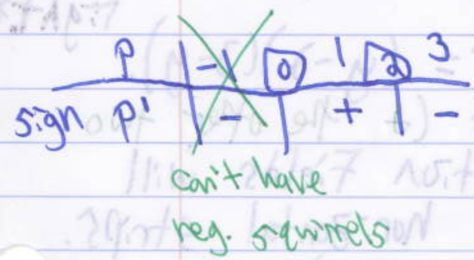
e.g.) $a = 2$, $b = 1$:

$$P = \frac{2P_0}{P_0 + (2 - P_0)e^{-2t}}.$$

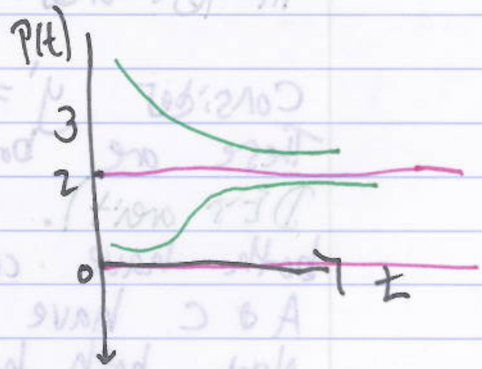
3. Suppose that the population $p$ (in thousands) of squirrels in Hamilton can be modelled by the DE $\frac{dp}{dt} = p(2-p)$.

ⓐ If the initial population of squirrels is 3000, what can you say about the long-term behaviour of the squirrel population?

$p(2-p) = 0 \iff p = 0$ or $p = 2$.








Squirrel population will tend to 2000 in the long-term.

ⓑ Can a population of 1000 ever decline to 500? Explain.

No, b/c solution curves strictly increasing b/w $0 < p < 2$, so if we begin w/ 1000 it'll never decrease to 500.

ⓒ Can a population of 1000 ever increase to 3000? Explain.

No, b/c a solution curve starting at $p=1$ can't cross the constant solution $p=2$, so it will tend to 2000, but can never reach 3000.

4. Consider the IVP $y' = 2x - 3y + 1$, $y(1) = 5$. Find an approximation of $y(1.2)$ using Euler's method with a step size of $h = 0.1$.

$$y_{n+1} = y_n + \frac{1}{10} F(x_n, y_n). \quad x_n = x_0 + nh.$$

Here $F(x,y) = 2x - 3y + 1$. $x_0 = 1$, $y_0 = 5$.

$\underline{n=1}$: $y_1 = y_0 + \frac{1}{10} F(x_0, y_0) = 5 + \frac{1}{10} F(1,5)$

$= 5 + \frac{1}{10}(2 - 15 + 1) = 5 + \frac{1}{10}(-12) = \frac{50}{10} - \frac{12}{10}$

$= \frac{38}{10} = \frac{19}{5}$. So, $x_1 = x_0 + h = 1 + \frac{1}{10} = \frac{11}{10}$.

$(x_1, y_1) = (\underbrace{\frac{11}{10}}_{1.1}, \underbrace{\frac{19}{5}}_{3.8})$.

$\underline{n=2}$: $x_2 = x_0 + 2\left(\frac{1}{10}\right) = \frac{10}{10} + \frac{2}{10} = \frac{12}{10} = 1.2$.

$y_2 = y_1 + \frac{1}{10} F(x_1, y_1) = \frac{19}{5} + \frac{1}{10}\left(\frac{22}{10} - \frac{57}{5} + 1\right)$

$= \frac{19}{5} + \frac{1}{10}\left(\frac{11}{5} - \frac{52}{5}\right) = \frac{19}{5} + \frac{1}{10}\left(-\frac{41}{5}\right)$

$= \frac{190}{50} - \frac{41}{50} = \frac{149}{50} = 2.98$. $(x_2, y_2) = (1.2, 2.98)$.

$\therefore y(1.2) \approx 2.98$.