Math 2203 - Tutorial #12

1. (a) Find the minimum radius of convergence of a power series solution of $(x^2-25)y'' + 2xy' + y = 0$ about the ordinary point: (i) $x=0$, (ii) $x=1$.

Recall: A point x_0 is said to be an <u>ordinary point</u> of the DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ if both $P(x)$ & $Q(x)$ in standard form $y'' + P(x)y' + Q(x)y = 0$ are analytic at x_0. A point that is <u>not</u> an ordinary point is called a <u>singular point</u>.

Recall: A function f is <u>analytic</u> at a point x_0 if it can be represented by a power series in $x-x_0$ with a positive radius of convergence.

<u>i.e.</u> It can be written in the form $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ & $\exists$ an interval R s.t. for all pts $a \in R$ we have that $\lim_{N\to\infty} \sum_{n=0}^{N} c_n(x-a)^n$ exists.

<u>Fact:</u> If $a_2(x), a_1(x)$ & $a_0(x)$ are polynomials w/ no common factors, then $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ has $x=x_0$ as an ordinary pt iff $a_2(x_0) \neq 0$.

Recall: <u>Existence of Power Series Solution:</u> If $x=x_0$ is an ordinary pt of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then we can always find 2 lin. ind. power series solutions centered at x_0 with minimum radius of converge R, where R is the distance to the closest singular point.

<u>i.e.</u> $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$.

(i) <u>$x=0$:</u> Only singular pts are when $x^2-25=0 \Rightarrow x = \pm 5$.

$|\pm 5 - 0| = 5 \Rightarrow \boxed{R=5}$.

(ii) <u>$x=1$:</u> $|5-1| = 4 \Rightarrow \boxed{R=4}$.

(b) Find the minimum radius of convergence of a power series solution of $(x^2-2x+10)y''+xy'-4y=0$ about the ord. pt. $x=1$.

$$x^2-2x+10=0 \Rightarrow x=\frac{2\pm\sqrt{4-40}}{2}=1\pm 3i.$$

$$R=\sqrt{(1-1)^2+(0-3)^2}=3 \Rightarrow R=3.$$

2. (a) Find the first 5 nonzero terms in the general solution to $(1+x^2)y''-y'+y=0$.

(b) Explain why this is the general solution. i.e., identify 2 linearly independent solutions.

$$y=\sum_{n=0}^{\infty}c_n x^n,\quad y'=\sum_{n=1}^{\infty}c_n n x^{n-1},\quad y''=\sum_{n=2}^{\infty}c_n n(n-1)x^{n-2}.$$

$$0=(1+x^2)y''-y'+y=(1+x^2)\sum_{n=2}^{\infty}c_n n(n-1)x^{n-2}$$

$$-\sum_{n=1}^{\infty}c_n n x^{n-1}+\sum_{n=0}^{\infty}c_n x^n$$

$$=\underbrace{\sum_{n=2}^{\infty}c_n n(n-1)x^{n-2}}_{K=n-2,\ n=K+2}+\underbrace{\sum_{n=2}^{\infty}c_n n(n-1)x^{n}}_{K=n}-\underbrace{\sum_{n=1}^{\infty}c_n n x^{n-1}}_{K=n-1,\ n=K+1}+\underbrace{\sum_{n=0}^{\infty}c_n x^n}_{K=n}$$

$$=\sum_{K=0}^{\infty}c_{K+2}(K+2)(K+1)x^K+\sum_{K=2}^{\infty}c_K K(K-1)x^K-\sum_{K=0}^{\infty}c_{K+1}(K+1)x^K+\sum_{K=0}^{\infty}c_K x^K$$

$$=(2c_2+6c_3x)-(c_1+2c_2x)+(c_0+c_1x)$$

$$+\sum_{K=2}^{\infty}\left(c_{K+2}(K+2)(K+1)+c_K[K(K-1)+1]-c_{K+1}(K+1)\right)x^K$$

$-\frac{c_0}{6} + \frac{-3c_1+3c_0}{6}$

$$c_2 = \frac{c_1 - c_0}{2} \qquad c_3 = \frac{-c_1 + 2c_2}{6} = \frac{-c_1 + c_1 - c_0}{6} = \frac{-1}{6}c_0$$

$$\exists\ 2c_2 - c_1 + c_0 = 0 \ \ \& \ \ 6c_3 - 2c_2 + c_1 = 0$$

$$\& \quad \underbrace{c_{k+2} = \frac{(k+1)\,c_{k+1} - [k(k-1)+1]\,c_k}{(k+2)(k+1)}, \quad k \geq 2}_{\text{recurrance relation}}$$

$$c_4 = \frac{3c_3 - 3c_2}{12} = \frac{c_3 - c_2}{4} = \frac{-\frac{1}{6}c_0 - \frac{c_1 + c_0}{2}}{4} = \frac{-3c_1 + 2c_0}{24}$$

$\therefore$ The First terms in the general solution are:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$= c_0 + c_1 x + \frac{(c_1 - c_0)}{2} x^2 + \left(-\frac{1}{6}c_0\right) x^3 + \frac{-3c_1 + 2c_0}{24} x^4 + \ldots$$

$\boxed{b}$ Choosing $c_0 = 0$ & $c_1 = 1$ we have:

$$y_1 = x + \frac{1}{2}x^2 - \frac{3}{24}x^4 + \ldots$$

Choosing $c_0 = 1$ & $c_1 = 0$ we have:

$$y_0 = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{2}{24}x^4 + \ldots$$

y_1 & y_2 are linearly independent solutions to a 2^{nd} order DE, so by def^n, the general solution is:

$y = c_0 y_0 + c_1 y_1$, which is what we found in $\boxed{a}$.

3. $\boxed{a}$ Find ~~a~~ the first 4 terms of a general solution for $2y'' + xy' + y = 0$.

$$0 = 2y'' + xy' + y$$

$$= 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n$$

$$= \underbrace{\sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2}}_{\underbrace{k=n-2}_{n=k+2}} + \underbrace{\sum_{n=1}^{\infty} c_n n x^n}_{n=k} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{n=k}$$

$$= \sum_{k=0}^{\infty} 2(k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} c_k k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= 2(2)(1)c_2 + c_0 + \sum_{k=1}^{\infty}\left[2(k+2)(k+1)c_{k+2} + (k+1)c_k\right]x^k$$

$$\Rightarrow \underbrace{c_0 + 4c_2 = 0}_{c_2 = \frac{-c_0}{4}} \quad \& \quad \underbrace{c_{k+2} = \frac{-(k+1)c_k}{2(k+2)(k+1)} = \frac{-c_k}{2(k+2)}}_{c_3 = \frac{-c_1}{2(3)} = \frac{-c_1}{6}}$$

$$\therefore\ y = c_0 + c_1 x + \left(-\frac{c_0}{4}\right)x^2 + \left(\frac{-c_1}{6}\right)x^3 + \cdots.$$

$\boxed{b}$ ~~Solve~~ [find first 2 terms] of the IVP $2y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

$$y(0) = 1 \Rightarrow \sum_{n=0}^{\infty} c_n x^n \Big|_{x=0} = 1 \Rightarrow c_0 = 1.$$

$$y'(0) = 0 \Rightarrow \sum_{n=1}^{\infty} c_n n x^{n-1} \Big|_{x=0} = 0 \Rightarrow c_1 = 0.$$

$$\therefore\ y = 1 - \frac{1}{4}x^2 + \cdots.$$