Math 2203 - Tutorial #11

1. Use the Laplace Transform to solve the IVP
$y'' + y = \sqrt{2}\sin(\sqrt{2}t)$, $y(0)=10$, $y'(0)=0$.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sqrt{2}\sin(\sqrt{2}t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + [Y(s)] = \sqrt{2}\,\mathcal{L}\{\sin(\sqrt{2}t)\}$$

$$(s^2+1)Y(s) - 10s = \sqrt{2}\cdot\frac{\sqrt{2}}{s^2+2}$$

$$Y(s) = \frac{2}{(s^2+2)(s^2+1)} + \frac{10s}{(s^2+1)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{2}{(s^2+2)(s^2+1)}\right\} + \underbrace{10\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{10\cos t}$$

$$\frac{2}{(s^2+2)(s^2+1)} = \frac{Bs+C}{(s^2+1)} + \frac{Ds+E}{(s^2+2)}$$

$$\Rightarrow 2 = (Bs+C)(s^2+2) + (Ds+E)(s^2+1)$$

$$\Rightarrow 2 = Bs^3 + 2Bs + Cs^2 + 2C + Ds^3 + Ds + Es^2 + E$$

$$\Rightarrow 2 = \underbrace{(B+D)}_{=0}s^3 + \underbrace{(C+E)}_{=0}s^2 + \underbrace{(2B+D)}_{=0}s + \underbrace{(2C+E)}_{=2}$$

$$\Rightarrow B=-D,\quad \underbrace{C=-E}_{C=2},\quad \underbrace{2B=-D}_{\substack{-2D+D=0\\ -D=0\\ D=0 \Rightarrow B=0}},\quad \underbrace{-2E+E=2}_{E=-2}$$

So, $$\frac{2}{(s^2+2)(s^2+1)} = \frac{2}{(s^2+1)} - \frac{2}{(s^2+2)}.$$

$\therefore \; y(t) = 10\cos t + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$

$\Rightarrow y(t) = 10\cos t + 2\sin t - \frac{2}{\sqrt{2}}\sin(\sqrt{2}\,t).$

2. Find $\mathcal{L}\{te^{-6t}\}$.

$\mathcal{L}\{te^{-6t}\} = \mathcal{L}\{t\}\big|_{s\to s+6} = \frac{1}{s^2}\Big|_{s\to s+6} = \frac{1}{(s+6)^2}.$

3. Find $\mathcal{L}\{\sin t\, \mathcal{U}(t-\pi/2)\}$.

$\mathcal{L}\{\sin t\, \mathcal{U}(t-\frac{\pi}{2})\} = e^{-\frac{\pi}{2}s}\,\mathcal{L}\{\underbrace{\sin(t+\tfrac{\pi}{2})}_{\cos t}\} = e^{-\frac{\pi}{2}s}\,\frac{s}{s^2+1}.$

4. Solve $y'+y = f(t)$, $y(0)=0$, for $Y(s)=\mathcal{L}\{y\}$, where $f(t)=\begin{cases}1, & 0\le t<1\\ -1, & t\ge 1.\end{cases}$

By def'n, $\mathcal{U}(t-a) = \begin{cases}0 & t<a\\ 1 & t\ge a.\end{cases}$

So, $f(t) = 1 - 2\,\mathcal{U}(t-1).$

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{1 - 2\,\mathcal{U}(t-1)\}$

$[sY(s) - y(0)] + Y(s) = \frac{1}{s} - 2\frac{e^{-s}}{s}$

$(s+1)Y(s) = \frac{1-2e^{-s}}{s} \Rightarrow \boxed{Y(s) = \frac{1-2e^{-s}}{s(s+1)}}.$

Could solve for $y(t)$ by applying $\mathcal{L}^{-1}$ to both sides:

$y(t) = \mathcal{L}^{-1}\left\{\frac{1-2e^{-st}}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{e^{-st}}{s(s+1)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} - 2\,\mathcal{U}(t-1)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}\Bigg|_{t\to t-1}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow 1 = A(s+1) + Bs \Rightarrow A+B=0 \;\&\; A=1$$
$$\Rightarrow B=-1.$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} - 2\,\mathcal{U}(t-1)\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}\Bigg|_{t\to t-1}$$

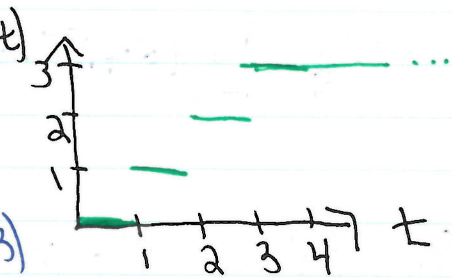
$$= 1 - e^{-t} - 2\,\mathcal{U}(t-1)\left[1 - e^{-t}\right]\Big|_{t\to t-1}$$

$$= 1 - e^{-t} - 2\,\mathcal{U}(t-1)\left[1 - e^{-(t-1)}\right].$$

5. Find $\mathcal{L}^{-1}\left\{\frac{6}{(s-5)^4}\right\}$.

$$= \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} e^{5t} = \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} e^{5t} = e^{5t}\, t^3.$$

6. Find $\mathcal{L}\{F(t)\}$, where:



$$F(t) = \mathcal{U}(t-1) + \mathcal{U}(t-2) + \mathcal{U}(t-3)$$

$$\Rightarrow \mathcal{L}\{F(t)\} = \frac{e^{-s} + e^{-2s} + e^{-3s}}{s}.$$

7. Evaluate $\mathcal{L}\left\{\int_0^t \tau \sin\tau \, d\tau\right\}$.

<u>Recall</u>: • $f*g = \int_0^t f(\tau) g(t-\tau)\, d\tau$ <u>convolution</u>.

• $\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.

Here, let $f(t) = t\sin t$, $g(t) = 1$. Then $f*g = \int_0^t \tau\sin\tau\, d\tau$,

so $\mathcal{L}\left\{\int_0^t \tau \sin\tau\, d\tau\right\} = \mathcal{L}\{t\sin t * 1\}$

$= \mathcal{L}\{t \sin t\}\, \mathcal{L}\{1\}$

<u>Recall</u>: $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}$

$= \left[(-1)^1 \frac{d}{ds} \mathcal{L}\{\sin t\}\right] \frac{1}{s} = \frac{-1}{s}\left[\frac{d}{ds} \frac{1}{s^2+1}\right]$

$= \frac{-1}{s}\left[-(s^2+1)^{-2} \cdot 2s\right] = \frac{2}{(s^2+1)^2}$.

8. Solve $y' - 3y = \delta(t-2)$, $y(0) = 0$.

$\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2)\}$

<u>Recall</u>: $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$.

$sY(s) - y(0) - 3Y(s) = e^{-2s}$

$Y(s) = \frac{e^{-2s}}{s-3} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-3}\right\}$

$$= \mathcal{U}(t-2)\, \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}\Big|_{t \to t-2}$$

$$= \mathcal{U}(t-2)\, e^{3(t-2)}.$$

9. Use the Laplace transform to solve the system:

$$\begin{cases} x' = -x + y \\ y' = 2x \end{cases}, \quad x(0) = 0,\ y(0) = 1.$$

$$\begin{cases} \mathcal{L}\{x'\} = -\mathcal{L}\{x\} + \mathcal{L}\{y\} \\ \mathcal{L}\{y'\} = 2\mathcal{L}\{x\} \end{cases} \Rightarrow \begin{cases} sX(s) - x(0) = -X(s) + Y(s) \\ sY(s) - y(0) = 2X(s) \end{cases}$$

$$\Rightarrow \begin{cases} (s+1)X(s) - Y(s) = 0 \\ -2X(s) + sY(s) = 1 \end{cases} \Rightarrow \begin{bmatrix} s+1 & -1 \\ -2 & s \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{1}{s(s+1)-2} \begin{bmatrix} s & 1 \\ 2 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+s-2} \begin{bmatrix} 1 \\ s+1 \end{bmatrix}$$

$$\Rightarrow X(s) = \frac{1}{(s+2)(s-1)} \quad \& \quad Y(s) = \frac{s+1}{(s+2)(s-1)}$$

Partial fractions $= \frac{-1}{3(s+2)} + \frac{1}{3(s-1)}$ Partial fractions $= \frac{1}{3(s+2)} + \frac{2}{3(s-1)}$

$$\Rightarrow x(t) = -\tfrac{1}{3} e^{-2t} + \tfrac{1}{3} e^{t} \quad \& \quad y(t) = \tfrac{1}{3} e^{-2t} + \tfrac{2}{3} e^{t}.$$

10. Find $\mathcal{L}\{t e^{2t} \sin(6t)\}$.

$= -\frac{d}{ds} \mathcal{L}\{e^{2t}\sin(6t)\} = -\frac{d}{ds}\left[\mathcal{L}\{\sin(6t)\}\Big|_{s\to s-2}\right]$

$= -\frac{d}{ds}\left[\frac{6}{s^2+36}\Big|_{s\to s-2}\right] = -\frac{d}{ds}\frac{6}{(s-2)^2+36}$

$= 6\left[((s-2)^2+36)^{-2} \cdot 2(s-2)\right] = \frac{12(s-2)}{(s^2-4s+40)^2}$.